METHODS OF SUBOPTIMAL CONTROL FOR LINEAR REGULATORS USING OUTPUT FEEDBACK

David Thomas Cowdrill

Gaylord
SHELF BINDER
Syracuse, N. Y.
Stockton, Calif.

LIBRARY
NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIF. 93940

United States
Naval Postgraduate School

THESIS

METHODS OF SUBOPTIMAL CONTROL
FOR
LINEAR REGULATORS USING OUTPUT FEEDBACK

by

David Thomas Cowdrill

Thesis Advisor: D. E. Kirk

March 1971

Approved for public release; distribution unlimited.

T139343

Methods of Suboptimal Control
for
Linear Regulators Using Output Feedback

by

David Thomas Cowdrill
Lieutenant Commander, United States Navy
B.S., University of Notre Dame, 1958

Submitted in partial fulfillment of the
requirements for the degree of

ELECTRICAL ENGINEER

from the

NAVAL POSTGRADUATE SCHOOL
March 1971

C 7̄56955
c. 1

LIBRARY
NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIF. 93940

ABSTRACT

Methods are discussed for optimal and suboptimal control of linear regulator systems employing controllers which use only accessible states and which can be easily realized. The conditions required for stability of such systems are shown and an algorithm is introduced for determining the elements of a stabilizing constant-gain matrix. This matrix provides an initial point for a technique which suboptimizes the performance of systems with infinite-time performance measures. The concept of partial canonic state feedback is introduced in which a Luenberger observer obtains missing canonic state information which is combined with the plant output to form a feedback vector. Also given is an extension of an existing algorithm for determining piecewise-constant gains to include switching times as variable parameters. Examples are given to numerically illustrate the concepts and results.

TABLE OF CONTENTS

ACKNOWLEDGEMENT

The author wishes to express his sincere appreciation to Professor Donald E. Kirk for his good counsel and encouragement and to Dr. Sidney Yakowitz for his kind assistance.

I. INTRODUCTION

This thesis is an investigation of suboptimal control of linear regulator systems. It delves into various approaches which ease the restrictions on the optimal controller realization, and into the stability problems which arise. Two variations of a previous solution method and one new method which seems very promising are introduced.

Chapter II defines the linear regulator problem, stating results for the optimal solution due to Kalman and describing the major practical difficulties with implementation of an optimal controller. The concepts of controllability and observability, which play a major role in the sequel, are defined.

Chapter III describes control methods which alleviate one or more of the implementation difficulties. For systems with only outputs measurable, several methods are described which achieve optimal performance, though greater complexity is required because of the missing states. Other methods provide performance less than optimal in order to gain simplicity of design; this latter approach is pursued in later chapters.

Chapter IV discusses the stability of linear time-invariant systems with output feedback. Since Kalman's canonic decomposition is fundamental to this problem as well as to Chapter VI, it is explained in detail. Conditions for stabilizability for non-dynamic controllers are discussed and extended to controllers using dynamic estimators. A straightforward and

effective search method for finding an output-feedback gain matrix for stability is presented.

Chapters V and VI are concerned with two different methods for finding suboptimal controls. Ozer's method [O-1] is a search technique minimizing an auxiliary performance measure. In Chapter V the technique is extended to finding optimum switching times and gains for piecewise-constant feedback. A variation of this method is also used to find gains for output feedback when an infinite-time performance measure is specified. Chapter VI presents a new method called "partial canonic state feedback" which utilizes all of the information present in the output feedback control.

A fifth-order system describing a string of three moving vehicles is utilized throughout as a numerical example, providing continuity and the capability for comparison of different control methods.

II. THE LINEAR REGULATOR PROBLEM

A. THE LINEAR REGULATOR PROBLEM----OPTIMAL CONTROL

1. Introduction

The linear regulator is much studied in control theory, both classical and modern. The equations below describe a linear dynamic plant whose states are to be brought and kept near zero. This concept has many applications in engineering problems, where often the states may be deviations of an actual condition from a desired one. The work of R. E. Kalman (K-2) has provided a complete theory and closed form solution for the minimum cost and the optimal control which provides it.

2. The Finite Time Linear Regulator Problem

For the plant described[1] by

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}(t)\ \underset{\sim}{x} + \underset{\sim}{B}(t)\ \underset{\sim}{u}, \tag{1}$$

where $\underset{\sim}{x}$ is an n-vector of states of the plant and $\underset{\sim}{u}$ is an m-vector of plant input controls, it is desired to find the control $\underset{\sim}{u}(\cdot)$ which minimizes the performance measure or cost

$$J(\underset{\sim}{x}(0),\underset{\sim}{u}(\cdot)) = \tfrac{1}{2}\underset{\sim}{x}^T(t)\underset{\sim}{H}\underset{\sim}{x}(t) + \tfrac{1}{2}\int_0^{t_f} (\underset{\sim}{x}^T(t)\underset{\sim}{Q}(t)\underset{\sim}{x}(t) + \underset{\sim}{u}^T(t)\underset{\sim}{R}(t)\underset{\sim}{u}(t))\,dt\ . \tag{2}$$

[1]The time dependence of $\underset{\sim}{x}(t)$, $\underset{\sim}{u}(t)$ and $\underset{\sim}{y}(t)$ will normally be suppressed when writing the state equations.

$R(t)$ is a real positive definite matrix, $Q(t)$ and H are real positive semi-definite matrices, and t_f is specified and finite.

The optimal control (in the sense that it uniquely minimizes $J(x(0),\ u(\cdot))$) is a linear feedback of all states with time-varying gains,

$$u_*(t) = - F^*(t)\ x(t) \tag{3}$$

where

$$F^*(t) = R^{-1}(t)\ B^T(t)\ K(t)\ . \tag{4}$$

$K(t)$ is the n x n unique symmetric solution of the Riccati equation

$$\begin{aligned}\dot{K}(t) = &- K(t)\ A(t) - A^T(t)\ K(t) \\ &+ K(t)\ B(t)\ R^{-1}(t)\ B^T(t)\ K(t) - Q(t)\end{aligned} \tag{5}$$

that satisfies the boundary condition

$$K(t_f) = H\ .$$

Further, the optimal cost is given by $J^* = \frac{1}{2}\, x^T(0)\ K(0)\ x(0)$.

3. The Infinite Time-Linear Regulator Problem

For a completely controllable plant described by

$$\dot{x} = A\ x + B\ u \tag{6}$$

where A and B are constant matrices, it is desired to find the control $u\ (\cdot)$ which minimizes the cost

$$J\ (x(0),\ u(\cdot)) = \frac{1}{2}\int_0^\infty (x^T(t)\ Q\ x(t) + u^T(t)\ R\ u(t))dt \tag{7}$$

where $\underset{\sim}{R}$ is a real positive definite matrix and $\underset{\sim}{Q}$ is a real positive semi-definite matrix. The added requirement of complete controllability[2] is important.

The optimal control is the linear feedback of all states with a constant gain:

$$\underset{\sim}{u}_*(t) = - \underset{\sim}{F}^* \underset{\sim}{x}(t) , \tag{8}$$

where

$$\underset{\sim}{F}^* = \underset{\sim}{R}^{-1} \underset{\sim}{B}^T \underset{\sim}{K}_{ss} \tag{9}$$

and $\underset{\sim}{K}_{ss}$ is the unique symmetric steady-state solution of Riccati equation (5), with the initial condition $\underset{\sim}{K}(\infty) = \underset{\sim}{0}$. $\underset{\sim}{K}_{ss}$ satisfies the algebraic equation,

$$- \underset{\sim}{K}_{ss} \underset{\sim}{A} - \underset{\sim}{A}^T \underset{\sim}{K}_{ss} + \underset{\sim}{K}_{ss} \underset{\sim}{B} \underset{\sim}{R}^{-1} \underset{\sim}{B}^T \underset{\sim}{K}_{ss} - \underset{\sim}{Q} = \underset{\sim}{0} . \tag{10}$$

As before, the optimal cost is $J^* = \frac{1}{2} \underset{\sim}{x}^T(0) \underset{\sim}{K}_{ss} \underset{\sim}{x}(0)$. The derivation of the above results and further discussion may be found in textbooks on optimal control theory, for instance [A-2, K-7].

4. The Meaning of the Term "Optimal"

In the literature, the term "optimal" is sometimes applied to systems which are different from the original plant. Ferguson and Rekasius [F-1], for example, propose a dynamic controller[3] which has q states that are derivatives of a

---

[2] The definition of controllability and references are given in part C of this chapter.

[3] A dynamic controller is one defined by a set of differential equations.

scalar control, $u(t)$. A control for the augmented system of order $n + q$ is found that minimizes a performance measure containing the controller states in addition to the plant states and controls. This control law is termed optimal, which is true for the augmented system. With regard to the original plant, however, the control law may not be optimal.

To avoid confusion of terminology, in this thesis the word "optimal" when applied to a system or control means the system described in sections 2 or 3 above. It is emphasized that in a discussion of suboptimal controls, the optimal system provides a useful reference point for comparison of other controllers.

B. MOTIVATION FOR SUBOPTIMAL CONTROL

1. Difficulty of Implementation

Though it is mathematically pleasing to have the solution for the optimal control for a linear regulator, many difficulties exist in its implementation:

a. All states must be accurately measured. In some cases this requirement may be satisfied by installation of sufficient instrumentation; in other situations the state information may be very difficult or expensive to obtain.

b. In the finite time problem, $n(n+1)/2$ time-varying gains must be pre-computed, stored, and applied to the n states in synchronization with problem time.

c. In the infinite time problem, the gains are constant but the existing theoretical results are based on the assumption that the system is completely controllable.

2. Suboptimal Approaches

In recent years researchers have investigated methods of achieving acceptably good performance relative to the optimal linear regulator with controllers that can be easily implemented. In most suboptimal methods the designer tries to alleviate one or more of the major practical difficulties described above. First, complete state availability is not assumed, and an output vector $\underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{x}$ is used as the controller input. Second, a state feedback controller is designed in which constant, piecewise constant, or other easily realizable gains are used instead of the optimal time-varying gains. A third approach is a combination of both, that is, an output feedback controller which approximates optimal gains. Reference [A-1] describes a somewhat different suboptimal approach to the linear regulator problem.

Of interest to the control engineer trying to find the best practical control for a specific problem are the questions: Does a certain suboptimal control method cause the performance measure to converge to J*, the optimal value, or, if not, what is the limiting value of the performance measure? For given constant $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ matrices, does there exist a constant or dynamic output feedback controller which will stabilize the system?

C. CONTROLLABILITY AND OBSERVABILITY

The concepts of controllability and observability are fundamental to the optimal and suboptimal linear regulator problem.

1. Definition of Controllability

As defined by Kalman [K-3]: A state $\underset{\sim}{x}_o$ of a linear stationary plant is said to be "controllable" if there exists a control signal u(t) defined over a finite interval $0 \leq t \leq t_1$, such that the state of the system at time t_1, $\underset{\sim}{\phi}(t_1; \underset{\sim}{x}_o, 0) = \underset{\sim}{0}$. In general the time t_1 will depend on $\underset{\sim}{x}_o$. If every state is controllable, the plant is said to be "completely controllable".

Essentially this means that in a linear completely controllable plant a control u(·) always exists which causes the plant to move from an arbitrary initial state $\underset{\sim}{x}_o$ to an arbitrary final state $\underset{\sim}{x}_{t_1}$ in a finite time, t_1.

Kalman's well-known test for complete controllability is: The linear, time-invariant system

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\,\underset{\sim}{x} + \underset{\sim}{B}\,\underset{\sim}{u}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{x} \qquad (11)$$

where $\underset{\sim}{x}$ is an n-vector, $\underset{\sim}{u}$ is an m-vector, and $\underset{\sim}{y}$ is a p-vector, is completely controllable if and only if rank $(\underset{\sim}{B} \;\; \underset{\sim}{A}\underset{\sim}{B} \;\; \cdots \;\; \underset{\sim}{A}^{n-1}\underset{\sim}{B}) = n$.

2. Definition of Observability

If by observing the output $\underset{\sim}{y}(t)$ during the finite time interval $[t_o, t_1]$ the state $\underset{\sim}{x}(t_o) = \underset{\sim}{x}_o$ can be determined, the

state $\underset{\sim}{x}_o$ is said to be observable at time t_o. If all states $\underset{\sim}{x}_o$ are observable for every t_o, the system is called completely observable, or simply observable [K-7].

A test for observability is given in Athans and Falb, [A-2]. The constant plant given by (11) is observable if and only if rank $(\underset{\sim}{C}^T \quad \underset{\sim}{A}^T \underset{\sim}{C}^T \cdots\cdot (\underset{\sim}{A}^T)^{n-1} \underset{\sim}{C}^T) = n$.

III. THE LINEAR REGULATOR--PRACTICAL REALIZATIONS

A. CONVERGENCE TO AN OPTIMAL SYSTEM

When designing a practical controller for a linear regulator system, an important preliminary question is whether it is possible to make its performance optimal using only the available states as controller inputs (Recall that optimal in this thesis refers to those systems described in sections II.A.2 and 3).

For the time-invariant system with infinite-time performance measure, it is known that optimal performance can be obtained when the system is completely controllable if all states are accessible. If this is not the case, the performance measure, if it converges, converges to $\hat{J}$, a number larger than J^*. Knowledge of $\hat{J}$ or its estimate is of interest to the designer, but this topic is not investigated here.

For those linear, time-invariant systems with an infinite time performance measure which cannot be made optimal, an even more basic question arises: can the system be made stable with output feedback? This question is pursued in detail in Chapter IV.

B. THE OPTIMAL LINEAR REGULATOR USING OUTPUTS

For completely controllable, completely observable systems there are two approaches to achieve an optimal system:

(a) reconstruct an estimate of the state vector, $\hat{\underset{\sim}{x}}$, for use with the optimal control law, $\underset{\sim}{u} = -\underset{\sim}{F}^* \hat{\underset{\sim}{x}}$;

(b) design a dynamic controller with the freedom to eliminate feedback from states not measured.

1. State Vector Reconstruction

Devices which produce an estimate of the state vector are the subject of a huge area of control theory. Only a brief description will be presented here of three estimators suitable as part of a linear regulator controller: the Kalman filter, the Luenberger observer and a method of Dellon and Sarachik.

The well-known Kalman filter [K-1], [K-6] is closely related to the linear regulator, being its dual. The objective of the Kalman design is to obtain an estimate of the state vector at a fixed time t_1 which is statistically optimal. Considering just the i^{th} element of $\underset{\sim}{x}(t_1)$, minimizing the variance of an estimate

$$E\ (x_i(t_1) - \beta_i)^2\quad ,$$

where β_i is the minimum variance estimate of $x_i(t_1)$, is equivalent to minimizing the quadratic performance measure

$$\underset{\sim}{r}^T(t_o)\underset{\sim}{P}_o\underset{\sim}{r}(t_o) + \int_{t_o}^{t_1} (\underset{\sim}{r}^T(t)\underset{\sim}{Q}(t)\underset{\sim}{r}(t) + \underset{\sim}{s}^T(t)\underset{\sim}{R}(t)\underset{\sim}{s}(t))dt\ .$$

$\underset{\sim}{R}(t)$ is positive definite for all t, $\underset{\sim}{Q}(t)$ and $\underset{\sim}{P}_o$ are positive semi-definite for all t and

$$\dot{\underset{\sim}{r}} = \underset{\sim}{A}^T(t)\ \underset{\sim}{r} + \underset{\sim}{C}^T(t)\ \underset{\sim}{s}\ , \tag{1}$$

satisfies a specified boundary condition

$$\underset{\sim}{r}(t_1) = \underset{\sim}{b}\ .$$

The Kalman filter consists of a system described by equation (1) with the control

$$\underset{\sim}{s}(t) = - \underset{\sim}{R}^{-1}(t)\ \underset{\sim}{C}(t)\ \underset{\sim}{P}(t)\ \underset{\sim}{r}(t) ,$$

where $\underset{\sim}{P}(t)$ is the solution of the Riccati matrix differential equation

$$\dot{\underset{\sim}{P}}(t) = + \underset{\sim}{P}(t)\underset{\sim}{A}^T(t) + \underset{\sim}{A}(t)\underset{\sim}{P}(t) - \underset{\sim}{P}(t)\underset{\sim}{C}^T(t)\underset{\sim}{R}^{-1}(t)\underset{\sim}{C}(t)\underset{\sim}{P}(t) + \underset{\sim}{Q}(t)$$

which satisfies the boundary condition,

$$\underset{\sim}{P}(t_o) = \underset{\sim}{P}_o .$$

Clearly, however, the Kalman filter is an n^{th} order system of considerable complexity.

The Luenberger observer [L-2], [L-3] is a deterministic estimator of order n-p for the n^{th}-order time invariant system. It is a linear time-invariant system using the p-dimensional plant output vector $\underset{\sim}{y}(t)$ and the control vector $\underset{\sim}{u}(t)$ as inputs. The observer system matrices are calculated by certain algebraic equations, with the n-p observer eigenvalues arbitrarily chosen.

A method of Dellon and Sarachik [D-1] produces state estimates of time-varying linear systems with an $(n-p)^{th}$-order system. The controller, however, is very complex with n (2n-p) time-varying gains in addition to the mn optimal gains, $\underset{\sim}{F}^*(t)$. For this reason, it is this writer's opinion that the method is of mathematical but not engineering interest.

In all of the above methods the device producing the state estimate, $\hat{\underset{\sim}{x}}(t)$, is followed by a controller using the

optimal gains to produce $\underset{\sim}{u}(t) = -\underset{\sim}{F}^*(t)\,\hat{\underset{\sim}{x}}$. When $\hat{\underset{\sim}{x}}(t) = \underset{\sim}{x}(t)$ the control $\underset{\sim}{u}(t)$ is optimal. Otherwise it is suboptimal, to a degree depending on the accuracy of $\hat{\underset{\sim}{x}}(t)$.

2. Controllers with Specified Structure

Two similar methods exist for controllers which eliminate feedback from states not measured. The Pearson and Ding method [P-1] is applicable to linear time-invariant plants with vector inputs, while the Ferguson and Rekasius [F-1] method is applicable to time-varying plants with scalar inputs. Each uses ν-1 derivatives of the plant input where ν is the observability index[4]. The control parameters are found which minimize a performance measure containing the states of the plant and the controller. Here the resulting system is called "optimal" and it is. But it is a different system than the original system. Like the method of Dellon and Sarachik, the Ferguson and Rekasius procedure suffers from much added complexity.

C. THE SUBOPTIMAL LINEAR REGULATOR

1. All States Available

Kleinman and Athans [K-8], [K-9] treat time-varying plants with all states available. The optimal feedback gains are approximated by a summation of constant matrices each multiplied by a scalar time-varying function. The number of terms

[4]The observability index of a system is defined as the least positive integer ν for which the observability matrix $\underset{\sim}{Q}_o = (\underset{\sim}{C}^T \quad \underset{\sim}{A}^T\underset{\sim}{C}^T \cdots\cdots (\underset{\sim}{A}^T)^{\nu-1} \quad \underset{\sim}{C}^T)$ has rank n. [L-2]

and the set of scalar time functions are chosen in advance while the matrix elements are computed by minimizing the trace of a "cost matrix" (defined later in section V.A). The sub-optimal controller gains depend on the initial state. Kleinman and Athans overcome this difficulty by assuming a uniform distribution of initial states on the unit hypersphere and in effect minimizing the expected value of the cost. It is proven that the performance approaches optimal under certain mathematical conditions.

2. Outputs Available

Simplified controllers using only the available outputs are the most appealing from the practical engineering aspect. They are suitable where the designer has decided that ease of implementation is as important to him as optimal performance. In general, system performance as quantified by the performance measure will not be optimal.

Ozer [O-1] has developed a method for calculating constant or piecewise-constant gains which minimize the maximum difference between suboptimal and optimal costs. Ozer applied the method to stationary and time-varying systems with a finite time interval of interest. Man [M-1] has used a similar method for systems with finite and infinite-time performance measures. In this thesis Ozer's method is extended to infinite time interval systems in a manner more efficient than Man's, and to piecewise-constant gains with a variable time interval.

With Ozer's method of minimizing the cost matrix, the designer can work with systems which were previously ignored

in modern control theory because they lacked the characteristics required for optimal performance. For instance, the performance of systems which are not controllable or observable may be suboptimized. By comparison with the optimal system the designer may judge whether or not he should advance to a higher level of controller complexity.

A time-invariant system which cannot be made optimal is equivalent to one which does not permit arbitrary (closed-loop) pole placement. It is useful to examine the system in a canonic form which reveals the limitations on output control of the system. This is done in the following chapter.

IV. TESTING FOR STABILITY

A. STABILITY

A pertinent question to ask about the linear regulator system,

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\,\underset{\sim}{x} + \underset{\sim}{B}\,\underset{\sim}{u}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{x} \tag{1}$$

$$J = \int_0^\infty (\underset{\sim}{x}^T(t)\underset{\sim}{Q}\,\underset{\sim}{x}(t) + \underset{\sim}{u}^T(t)\underset{\sim}{R}\,\underset{\sim}{u}\,(t))dt$$

where the required assumptions of section II.A.3 are met, is: Is it possible to stabilize the system with a controller using only the output vector $\underset{\sim}{y}$?

The controller may be dynamic or non-dynamic (constant gains). This question is crucial for the infinite-time suboptimal linear regulator; its answer is pursued with considerable, though not complete, success in this chapter. Valuable insight into the system is gained by using Kalman's canonic transformation [K-4]. A necessary condition for system stability due to Galperin and Dergacheva [G-1] follows in a straightforward manner. For non-dynamic controllers a computational method has been developed to determine stabilizing output gains. An example of this method is presented. For controllers that include dynamic elements (a Luenberger observer is considered), necessary conditions for stabilizability are developed.

B. A CANONIC TRANSFORMATION

Kalman [K-4] and Gilbert [G-2] showed that a linear dynamic system can be transformed into four subsystems:

1) controllable, not observable

2) controllable and observable

3) not controllable, not observable

4) not controllable, observable.

For the system (1) the nxnm controllability matrix, $\underset{\sim}{G} = (\underset{\sim}{B}\ \underset{\sim}{A}\underset{\sim}{B}\ \underset{\sim}{A}^2\underset{\sim}{B}\cdots\underset{\sim}{A}^{n-1}\underset{\sim}{B})$ is of rank $r \leq n$. Let $\mathcal{G}$ be the r-dimensional controllable subspace of R^n spanned by its basis, $\underset{\sim}{B}_G = (\underset{\sim}{b}_{G1}\ \underset{\sim}{b}_{G2}\ \underset{\sim}{b}_{G3}\cdots\underset{\sim}{b}_{Gr})$ where $\underset{\sim}{b}_{Gi}$ is an n-vector.

The observability matrix $\underset{\sim}{H} = (\underset{\sim}{C}^T\ \underset{\sim}{A}^T\ \underset{\sim}{C}^T(\underset{\sim}{A}^T)^2\ \underset{\sim}{C}^T\cdots(\underset{\sim}{A}^T)^{n-1}\underset{\sim}{C}^T)$ is of rank $k \leq n$. Let $\mathcal{H}$ be the k-dimensional observable subspace spanned by its basis, $\underset{\sim}{B}_H = (\underset{\sim}{b}_{H1}\ \underset{\sim}{b}_{H2}\cdots\underset{\sim}{b}_{Hk})$, where $\underset{\sim}{b}_{Hi}$ is an n-vector. Let $\mathcal{H}^{\perp}$ be the orthogonal complement[5] of $\mathcal{H}$ in R^n. Thus $\mathcal{H}^{\perp}$ is the n-k dimensional not-observable subspace of R^n.

Let $\mathcal{J}$ be $\mathcal{G} \cap \mathcal{H}^{\perp}$, so that $\mathcal{J}$ is the completely controllable, not observable subspace of dimension j with basis $\underset{\sim}{J}$. Let $\mathcal{L}$ be the r-j dimensional complement of $\mathcal{J}$ in $\mathcal{G}$; its basis is $\underset{\sim}{L}$.

---

[5] Given a subspace $\mathcal{Y}$ of R^n, there exists a subspace $\mathcal{Z}$ of R^n such that $\mathcal{Y} \oplus \mathcal{Z} = R^n$. That is: 1) for $\underset{\sim}{y} \in \mathcal{Y}$ and $\underset{\sim}{z} \in \mathcal{Z}$, $\underset{\sim}{y}$ and $\underset{\sim}{z}$ span R^n and 2) $\mathcal{Y} \cap \mathcal{Z} = \{0\}$. $\mathcal{Z}$ is called the complement of $\mathcal{Y}$ in R^n. If in addition $\langle \underset{\sim}{y}, \underset{\sim}{z} \rangle = 0$ for all $\underset{\sim}{y} \in \mathcal{Y}$ and $\underset{\sim}{z} \in \mathcal{Z}$, then $\mathcal{Z}$ is called the orthogonal complement of $\mathcal{Y}$. [W-2]

Let $\mathcal{M}$ be the n-k-j dimensional complement of $\mathcal{J}$ in $\mathcal{H}^{\perp}$; its basis is $\underset{\sim}{M}$.

Since $\mathcal{J} \oplus \mathcal{L} = \mathcal{G}$, $\mathcal{L}$ is a completely controllable, completely observable subspace; similarly, $\mathcal{M}$ is a not controllable, not observable subspace.

Now the state space, R^n, can be described as $R^n = \mathcal{J} \oplus \mathcal{L} \oplus \mathcal{M} \oplus \mathcal{N}$, where $\mathcal{N}$ is the complement of $\mathcal{J} \oplus \mathcal{L} \oplus \mathcal{M}$ in R^n, with basis $\underset{\sim}{N}$. Figure IV-1 illustrates the relationship in R^3.

The nxn matrix $\underset{\sim}{T} = (\underset{\sim}{J}\ \underset{\sim}{L}\ \underset{\sim}{M}\ \underset{\sim}{N})$ is then the matrix of a linear transformation which maps the basis vectors of a canonical space into the basis vectors of state space. $\underset{\sim}{T}$ is non-singular because it is comprised of the basis vectors of a direct sum of subspaces.[6]

Then

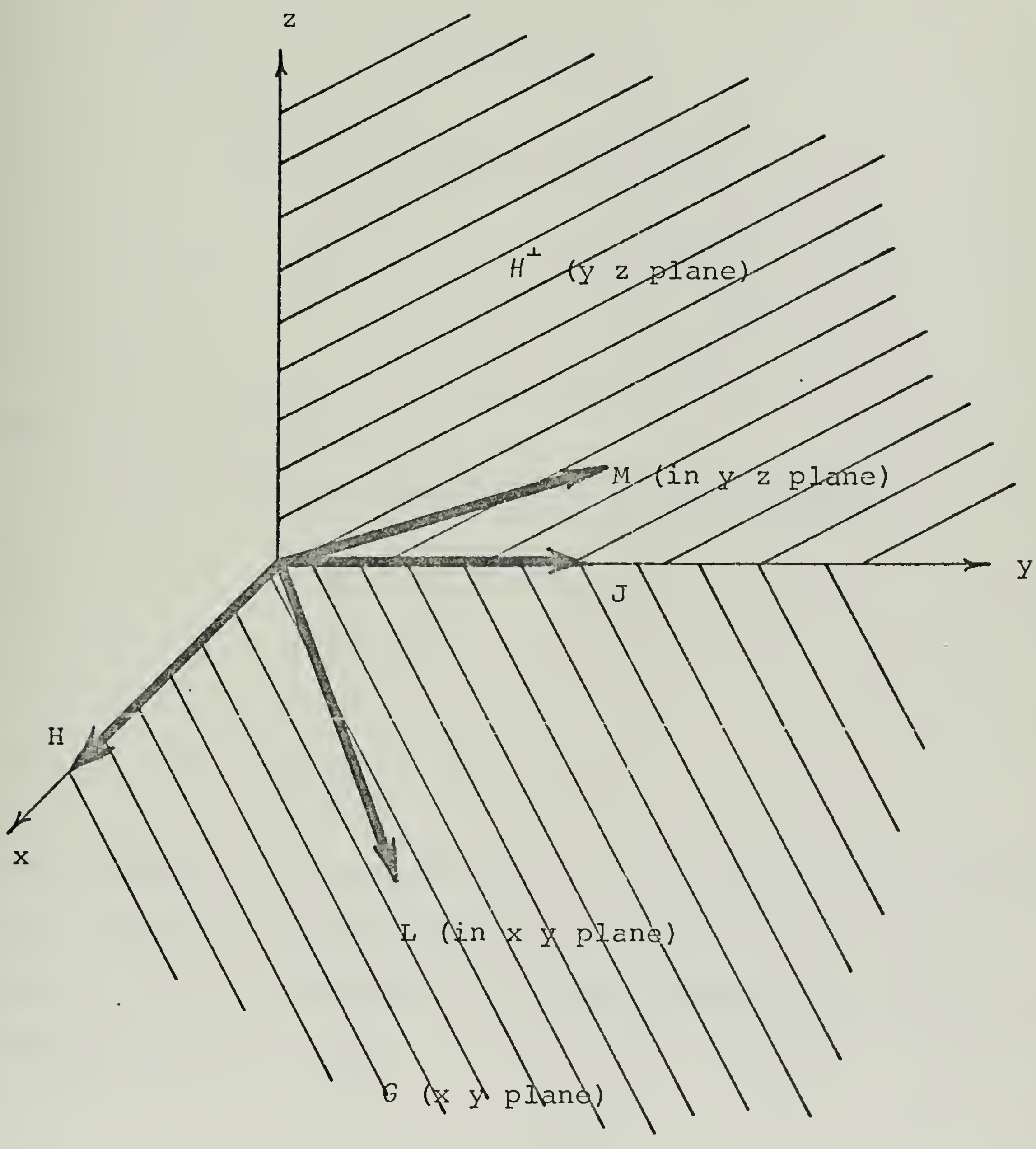
$$\begin{aligned}
\underset{\sim}{x} &= \underset{\sim}{T}\underset{\sim}{z} \\
\dot{\underset{\sim}{x}} &= \underset{\sim}{T}\dot{\underset{\sim}{z}} = \underset{\sim}{A}\ \underset{\sim}{T}\ \underset{\sim}{z} + \underset{\sim}{B}\ \underset{\sim}{u} \\
\dot{\underset{\sim}{z}} &= \underset{\sim}{T}^{-1}\ \underset{\sim}{A}\ \underset{\sim}{T}\ \underset{\sim}{z} + \underset{\sim}{T}^{-1}\ \underset{\sim}{B}\ \underset{\sim}{u} \\
\underset{\sim}{y} &= \underset{\sim}{C}\ \underset{\sim}{T}\ \underset{\sim}{z}
\end{aligned} \tag{2}$$

To evaluate $\underset{\sim}{T}^{-1}\ \underset{\sim}{A}\ \underset{\sim}{T}$, first consider $\underset{\sim}{A}\underset{\sim}{T} = \underset{\sim}{A}(\underset{\sim}{J}\ \underset{\sim}{L}\ \underset{\sim}{M}\ \underset{\sim}{N}) = (\underset{\sim}{A}\underset{\sim}{J}\ \underset{\sim}{A}\underset{\sim}{L}\ \underset{\sim}{A}\underset{\sim}{M}\ \underset{\sim}{A}\underset{\sim}{N})$. The key point is that $\mathcal{J}$, $\mathcal{G}$, and $\mathcal{H}^{\perp}$ are invariant[7] under $\underset{\sim}{A}$.

---

[6] If $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional subspaces of a vector space with bases $(x_1, x_2, \cdots x_m)$ and $(y_1, y_2, \cdots y_n)$ respectively, then: 1) dim $(\mathcal{X} \oplus \mathcal{Y})$ = dim $(\mathcal{X})$ + dim $(\mathcal{Y})$ = m+n and 2) a basis of $\mathcal{X} \oplus \mathcal{Y}$ is $(x_1, x_2, \cdots x_m, y_1 \cdots y_n)$. [W-2]

[7] A subspace $\mathcal{S}$ is said to be invariant under a transformation T if $T\underset{\sim}{s} \in \mathcal{S}$ for every $\underset{\sim}{s} \in \mathcal{S}$. That is: T maps $\mathcal{S}$ into itself.



N is zero-dimensional.

Fig. IV-1. Example of Canonic Relationships in R^3.

Thus

$$\underset{\sim}{A}\,\underset{\sim}{J} = (\underset{\sim}{A}\,\underset{\sim}{j}_1\ \underset{\sim}{A}\,\underset{\sim}{j}_2\ \cdots\ \underset{\sim}{A}\,\underset{\sim}{j}_j)$$

$$\underset{\sim}{A}\,\underset{\sim}{j}_k = \sum_{i=1}^{j} \underset{\sim}{b}_{ik}\,\underset{\sim}{j}_i = \underset{\sim}{J} \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{jk} \end{bmatrix}$$

and

$\underset{\sim}{A}\,\underset{\sim}{J} = \underset{\sim}{J}\,\underset{\sim}{D}_{11}$, where $\underset{\sim}{D}_{11}$ is the jxj matrix:

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} \\ b_{21} & \cdot & & \vdots \\ \vdots & & & \vdots \\ b_{j1} & \cdots & \cdots & b_{jj} \end{bmatrix} .$$

The subspace L is not invariant under $\underset{\sim}{A}$ but L is a subspace of G which is invariant under $\underset{\sim}{A}$, so that for $\underset{\sim}{\ell} \in L$, $\underset{\sim}{A}\,\underset{\sim}{\ell} \in G$ and $\underset{\sim}{A}\,\underset{\sim}{\ell}$ can be expressed as a linear combination of the basis vectors of G, $\underset{\sim}{J}$ and $\underset{\sim}{L}$.

$$\underset{\sim}{A}\,\underset{\sim}{L} = (\underset{\sim}{A}\,\underset{\sim}{\ell}_1\ \underset{\sim}{A}\,\underset{\sim}{\ell}_2\ \cdots\ \underset{\sim}{A}\,\underset{\sim}{\ell}_\ell)$$

$$\underset{\sim}{A}\,\underset{\sim}{\ell}_k = \sum_{i=1}^{j} c_{ik}\,\underset{\sim}{j}_i + \sum_{i=1}^{\ell} d_{ik}\,\underset{\sim}{\ell}_i$$

$$= \underset{\sim}{J} \begin{bmatrix} c_{1k} \\ c_{2k} \\ \vdots \\ c_{jk} \end{bmatrix} + \underset{\sim}{L} \begin{bmatrix} d_{1k} \\ d_{2k} \\ \vdots \\ d_{\ell k} \end{bmatrix}$$

and $\underset{\sim}{A}\,\underset{\sim}{L} = \underset{\sim}{J}\,\underset{\sim}{D}_{12} + \underset{\sim}{L}\,\underset{\sim}{D}_{22}$

$$\text{where } \underset{\sim}{D}_{12} = \begin{bmatrix} c_{11} & \cdots & c_{1\ell} \\ \vdots & & \vdots \\ c_{j1} & \cdots & c_{j\ell} \end{bmatrix}$$

$$\underset{\sim}{D}_{22} = \begin{bmatrix} d_{11} & \cdots & d_{1\ell} \\ \vdots & & \vdots \\ d_{\ell 1} & \cdots & d_{\ell\ell} \end{bmatrix} .$$

Following the same reasoning,

$$\underset{\sim}{A}\,\underset{\sim}{M} = \underset{\sim}{J}\,\underset{\sim}{D}_{13} + \underset{\sim}{M}\,\underset{\sim}{D}_{33}$$

$$\underset{\sim}{A}\,\underset{\sim}{N} = \underset{\sim}{J}\,\underset{\sim}{D}_{14} + \underset{\sim}{L}\,\underset{\sim}{D}_{24} + \underset{\sim}{M}\,\underset{\sim}{D}_{34} + \underset{\sim}{N}\,\underset{\sim}{D}_{44} .$$

Combining the above yields

$$\underset{\sim}{A}\,\underset{\sim}{T} = (\underset{\sim}{J}\underset{\sim}{D}_{11}\ \ \underset{\sim}{J}\underset{\sim}{D}_{12}+\underset{\sim}{L}\underset{\sim}{D}_{22}\ \ \underset{\sim}{J}\underset{\sim}{D}_{13}+\underset{\sim}{M}\underset{\sim}{D}_{33}\ \ \underset{\sim}{J}\underset{\sim}{D}_{14}+\underset{\sim}{L}\underset{\sim}{D}_{24}+\underset{\sim}{M}\underset{\sim}{D}_{34}+\underset{\sim}{N}\underset{\sim}{D}_{44})$$

$$= (\underset{\sim}{J}\ \underset{\sim}{L}\ \underset{\sim}{M}\ \underset{\sim}{N}) \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ 0 & D_{22} & 0 & D_{24} \\ 0 & 0 & D_{33} & D_{34} \\ 0 & 0 & 0 & D_{44} \end{bmatrix}$$

$$\text{and} \quad \underset{\sim}{T}^{-1}\,\underset{\sim}{A}\,\underset{\sim}{T} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ 0 & D_{22} & 0 & D_{24} \\ 0 & 0 & D_{33} & D_{34} \\ 0 & 0 & 0 & D_{44} \end{bmatrix}$$

where the elements of the matrices are matrices but the "~" symbol has been suppressed for clarity.

To evaluate $\underset{\sim}{T}^{-1}\underset{\sim}{B}$, it can be seen that $\underset{\sim}{B} = (\underset{\sim}{b}_1\ \underset{\sim}{b}_2 \cdots \underset{\sim}{b}_m)$ and $\underset{\sim}{b}_i \in G$, $i = 1, \ldots m$. Each $\underset{\sim}{b}_i$ is a linear combination of the basis vectors of G, so that

$\underset{\sim}{b}_k = \underset{\sim}{J}\ \hat{\underset{\sim}{b}}_{1k} + \underset{\sim}{L}\ \hat{\underset{\sim}{b}}_{2k}$ where $\hat{\underset{\sim}{b}}_{1k}$ is a j-vector and $\hat{\underset{\sim}{b}}_{2k}$ is an r-j vector.

Thus, $$\underset{\sim}{B} = (\underset{\sim}{J}\ \underset{\sim}{L}\ \underset{\sim}{M}\ \underset{\sim}{N}) \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \\ 0 \end{bmatrix}$$

where $\hat{\underset{\sim}{B}}_1 = (\hat{\underset{\sim}{b}}_{11}\ \hat{\underset{\sim}{b}}_{12} \cdots \hat{\underset{\sim}{b}}_{1m})$ and $\hat{\underset{\sim}{B}}_2$ likewise, and

$$\underset{\sim}{T}^{-1}\ \underset{\sim}{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \\ 0 \end{bmatrix} .$$

To evaluate $\underset{\sim}{C}\underset{\sim}{T}$, notice that vectors in the unobservable subspace $H^{\perp} = J \oplus M$ do not appear in the output vector $\underset{\sim}{y}$.

$$\underset{\sim}{C}\ \underset{\sim}{J} = \underset{\sim}{0} = \underset{\sim}{C}\ \underset{\sim}{M}$$

$$\underset{\sim}{C}\ \underset{\sim}{T} = (\underset{\sim}{0}\ \underset{\sim}{C}\ \underset{\sim}{L}\ \underset{\sim}{0}\ \underset{\sim}{C}\ \underset{\sim}{N})$$

$$\underset{\sim}{C}\ \underset{\sim}{T} = (\underset{\sim}{0}\ \hat{\underset{\sim}{C}}_1\ \underset{\sim}{0}\ \hat{\underset{\sim}{C}}_2),$$

where $\hat{\underset{\sim}{C}}_1 = \underset{\sim}{C}\ \underset{\sim}{L}$ and $\hat{\underset{\sim}{C}}_2 = \underset{\sim}{C}\ \underset{\sim}{N}$.

Finally the transformed system (2) can be written

$$\begin{bmatrix} \dot{\underset{\sim}{z}}_1 \\ \dot{\underset{\sim}{z}}_2 \\ \dot{\underset{\sim}{z}}_3 \\ \dot{\underset{\sim}{z}}_4 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ 0 & D_{22} & 0 & D_{24} \\ 0 & 0 & D_{33} & D_{34} \\ 0 & 0 & 0 & D_{44} \end{bmatrix} \begin{bmatrix} \underset{\sim}{z}_1 \\ \underset{\sim}{z}_2 \\ \underset{\sim}{z}_3 \\ \underset{\sim}{z}_4 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \\ 0 \end{bmatrix} \underset{\sim}{u}$$

$$\underset{\sim}{y} = (0 \ \ \hat{C}_1 \ \ 0 \ \ \hat{C}_2) \begin{bmatrix} \underset{\sim}{z}_1 \\ \underset{\sim}{z}_2 \\ \underset{\sim}{z}_3 \\ \underset{\sim}{z}_4 \end{bmatrix} \qquad (3)$$

where $\underset{\sim}{z}_1$ is a j dimensional vector of controllable but unobservable states,

$\underset{\sim}{z}_2$ is a r-j dimensional vector of controllable and observable states,

$\underset{\sim}{z}_3$ is a n-k-j dimensional vector of uncontrollable and unobservable states,

and $\underset{\sim}{z}_4$ is a k+j-r dimensional vector of uncontrollable but observable states.

Actual calculation of the canonic transformation $\underset{\sim}{T}$ was done using a decomposition method of Bhandarkar and Fahmy [B-1]. It consists of a sequence of computational steps based on the properties of the controllability and observability matrices.

C. NECESSARY CONDITIONS FOR STABILITY FOR A NON-DYNAMIC CONTROLLER

1. Development of Necessary Conditions

Following Galperin and Dergacheva [G-1] the control is constrained to be a constant-gain output feedback,

$$u(t) = - P\, y(t) = - P\, C\, x(t) = - P(0 \;\; \hat{C}_1 \;\; 0 \;\; \hat{C}_2)\; z(t) \, .$$

The resulting closed loop system

$$\dot{z} = \begin{bmatrix} D_{11} & D_{12} - \hat{B}_1 P \hat{C}_1 & D_{13} & D_{14} - \hat{B}_1 P \hat{C}_2 \\ 0 & D_{22} - \hat{B}_2 P \hat{C}_1 & 0 & D_{24} - \hat{B}_2 P \hat{C}_2 \\ 0 & 0 & D_{33} & D_{34} \\ 0 & 0 & 0 & D_{44} \end{bmatrix} z \qquad (4)$$

is shown in Figure IV-2.

The eigenvalues of the system matrix of (4) are the eigenvalues of D_{11}, $D_{22} - \hat{B}_2 P \hat{C}_1$, D_{33} and D_{44}. This may be seen by performing elementary column transformations on the matrix

$$D - \lambda I = \begin{bmatrix} D_{11} - \lambda I & D_{12} - \hat{B}_1 P \hat{C}_1 & D_{13} & D_{14} - \hat{B}_1 P \hat{C}_2 \\ 0 & D_{22} - \hat{B}_2 P \hat{C}_1 - \lambda I & 0 & D_{24} - \hat{B}_2 P \hat{C}_2 \\ 0 & 0 & D_{33} - \lambda I & D_{34} \\ 0 & 0 & 0 & D_{44} - \lambda I \end{bmatrix}$$

converting it to upper triangular form.

Necessary conditions for the system to be stabilizable are then:

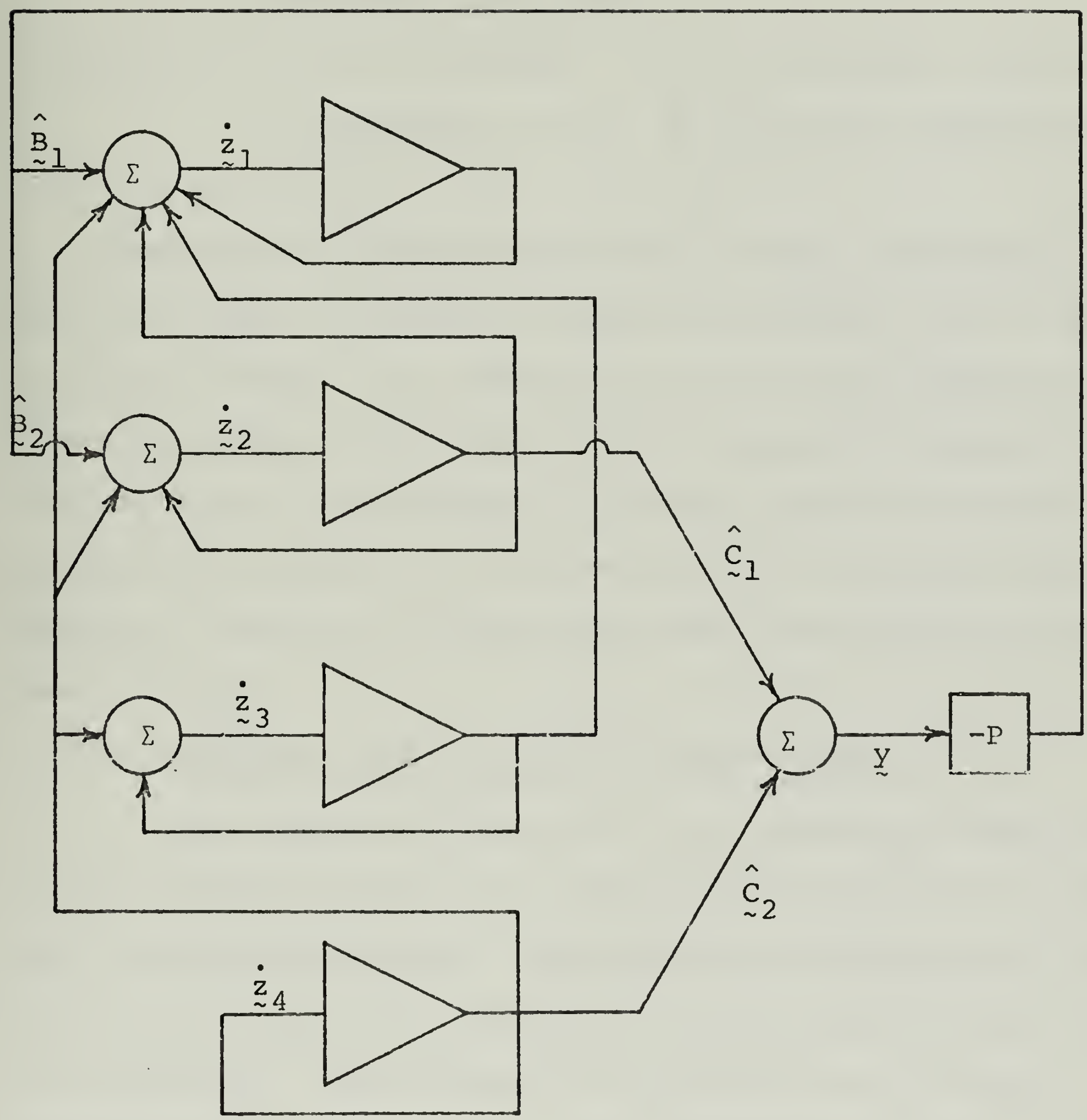


Fig. IV-2. Canonic Time-Invariant Linear System with Output Feedback.

Condition 1. $\underset{\sim}{D}_{11}$, $\underset{\sim}{D}_{33}$ and $\underset{\sim}{D}_{44}$ must have eigenvalues with negative real parts.

Condition 2. $\underset{\sim}{P}$ must be chosen (if it is possible) such that eigenvalues of $\underset{\sim}{D}_{22} - \underset{\sim}{\hat{B}}_2 \underset{\sim}{P} \underset{\sim}{\hat{C}}_1$ have negative real parts.

Condition 1 is useful and easy to apply; Condition 2 is more difficult. Methods of applying Condition 2 for a given $\underset{\sim}{P}$ are well-known. For unknown $\underset{\sim}{P}$ of one or two elements, the classical control techniques of root locus and parameter plane are suitable. For unknown $\underset{\sim}{P}$ of larger dimension finding stabilizing elements (if they exist) is a formidable task. A numerical method for accomplishing this is presented in the next section.

2. Finding a Stable $\underset{\sim}{P}$: A 9^{th} order Example

The existence problem for $\underset{\sim}{P}$ of necessary Condition 2 of the previous section is a surprisingly difficult one and has not yet been solved. Koenigsberg and Frederick in reference [K-10] explore the problem in detail, providing a list of 22 references. They develop an algorithm for finding a stabilizing $\underset{\sim}{P}$ if one exists. The algorithm is based on the sensitivity of the eigenvalues of the closed-loop system with respect to the elements of $\underset{\sim}{P}$.

In this section a much simpler, but still effective method is described. This method was used in section V.C for a fifth-order system with two and three available states. In this section a 9^{th}-order example problem of Koenigsberg and Frederick is also solved for a stabilizing $\underset{\sim}{P}$.

The method uses the pattern search procedure of Hooke and Jeeves [H-1, W-1]. The steps are:

1) Guess $\underset{\sim}{P}^{(o)}$

2) Use the pattern search library routine[8] to minimize an objective function whose value is determined as follows:

a) Calculate all eigenvalues[9] of $\underset{\sim}{A} + \underset{\sim}{B}\,\underset{\sim}{P}\,\underset{\sim}{C}$ for the current $\underset{\sim}{P}$.

b) Find the most positive real part of the eigenvalues.

c) Assign to the objective function the value of step b).

3) If and when the objective function has a value less than a specified negative real number, terminate the procedure.

The following problem was used as an example in [K-10]. The equations are those of a balanced beam on an electrically driven cart.

$$\dot{\underset{\sim}{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -20 & -4.2 & 0 & 4.45 & 12.5 & 0 & 100 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4.7 & 8.35 & 0 & -1.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 10.9 & 0 & 0 & -2.55 & -250 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 5.9 & 0 & 0 & -1.39 & 0 & 0 & -3700 & 0 \end{bmatrix} \underset{\sim}{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3.3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

---

[8] Subroutine DIRECT of the W. R. Church Computer Center, Naval Postgraduate School.

[9] Subroutine EIG3 of the same facility.

$$\underset{\sim}{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & .66 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & .66 & 0 & 1.2 \end{bmatrix} \underset{\sim}{x}$$

$$u = (K_1 \; K_2 \; K_3 \; K_4) \; \underset{\sim}{y}$$

Using initial values of $K_i = 0$, $i = 1, \ldots 4$ a step size of 50, and minimizing until the closed-loop eigenvalue locations are left of the vertical line at $a = -0.12$, where $z = a + j\,b$, a stable $\underset{\sim}{P}$ of (-50, -50, -450, -450) was found in 16 seconds on an IBM 360/67.

3. An Example Problem

In this section a fifth-order problem is tested for stability with constant-gain output feedback using the previously discussed methods. Later, in sections V.C. and VI.B. suboptimal controls for the same system are determined. The problem is one proposed and modeled by Levine and Athans [L-1] describing a string of moving vehicles whose position and velocity errors are to be regulated. Levine and Athans solve for the optimal state feedback gains which minimize an infinite time performance measure.

In the following the problem is modified so that not all states are available. Thus, stabilizability must be determined. Levine and Athans develop the following state equations for a string of three vehicles of unit mass and with unity drag coefficients.

$$\dot{\underset{\sim}{x}} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \underset{\sim}{x} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{u} \qquad (5)$$

where $\underset{\sim}{x} = \begin{bmatrix} \delta y_1 & (t) \\ \delta w_1 & (t) \\ \delta y_2 & (t) \\ \delta w_2 & (t) \\ \delta y_3 & (t) \end{bmatrix}$

δy_i is the velocity deviation with respect to the desired string velocity of the i^{th} vehicle. δw_i is the deviation from the desired value of the separation distance between the i^{th} and $(i+1)^{st}$ vehicles, and the drag coefficient is a term arising in the linearization of the original system equations.

The control $\underset{\sim}{u}$ is $\begin{bmatrix} \delta f_1 & (t) \\ \delta f_2 & (t) \\ \delta f_3 & (t) \end{bmatrix}$,

where δf_i is the force deviation from the force required to overcome drag at the string velocity for the i^{th} vehicle. The system is completely controllable.

It is desired to minimize the performance measure,

$$J = \int_0^\infty (\underset{\sim}{x}^T \underset{\sim}{Q} \underset{\sim}{x} + \underset{\sim}{u}^T \underset{\sim}{R} \underset{\sim}{u})dt \quad \text{where}$$

$$\underset{\sim}{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The $\underset{\sim}{Q}$ matrix is different from that of Levine and Athans. It has been made positive definite in order to solve for the suboptimal control in a later section.

First, measuring only the velocity deviations is considered. This is appealing at first thought because such a measurement scheme would require only accurate speedometers and a telemetry link. The output equation is

$$\underset{\sim}{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{x} \tag{6}$$

The observability matrix has rank = 3; therefore, the system is not observable.

Using the method of Bhandarkar and Fahmy [B-1] the canonic transformation, $\underset{\sim}{T}$ where $\underset{\sim}{x} = \underset{\sim}{T}\,\underset{\sim}{z}$ is found to be:

$$\underset{\sim}{T} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the canonic form of (5) and (6) is:

$$\dot{\underset{\sim}{z}} = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \underset{\sim}{z} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{u}$$

$$\underset{\sim}{y} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{\sim}{z}$$

Identification of the submatrices in the canonic system is done in the following way. The controllable subspace G has dimension 5, the unobservable subspace $H^{\perp}$ dimension 2, and thus J has dimension 2. Since L is the complement of J in G, L has dimension 3. Since M is the complement of J in $H^{\perp}$, it is zero dimensional. Thus $\underset{\sim}{D}_{11}$ is 2x2, the dimension of J; $\underset{\sim}{D}_{22}$ is 3x3, the dimension of L.

$$\underset{\sim}{D}_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{D}_{22} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Since $\underset{\sim}{D}_{11}$ has eigenvalues of 0, the system is not stabilizable with constant-gain output feedback.

Now if only distance deviations are measured, the output is

$$\underset{\sim}{y} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \underset{\sim}{x} \ . \qquad (7)$$

The observability matrix now has rank = 4, so again the system is not completely observable.

The canonic transformation $\underset{\sim}{T}$ for $\underset{\sim}{x} = \underset{\sim}{T}\underset{\sim}{z}$ is,

$$\underset{\sim}{T} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the canonic form of (5) and (7) is

$$\dot{\underset{\sim}{z}} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \underset{\sim}{z} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{u}$$

$$\underset{\sim}{y} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{z} \,. \qquad (8)$$

Analysis of the subspace dimensions as above shows that

$$\underset{\sim}{D}_{11} = (-1)$$

and

$$\underset{\sim}{D}_{22} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The system can be stabilized if a matrix $\underset{\sim}{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \\ p_{31} & p_{32} \end{bmatrix}$

can be found which causes

$$\underset{\sim}{D}_{22} - \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \\ p_{31} & p_{32} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

to have eigenvalues with negative real parts. Using the pattern search method described in section C.2., eight satisfactory $\underset{\sim}{P}$ matrices were found. These provided stable initial $\underset{\sim}{P}$ matrices for the suboptimization procedure of section V.B.

D. DYNAMIC CONTROLLERS

The difficulty of application of Condition 2 for stabilizability can be circumvented if dynamic controllers can be used. It is then possible to utilize an estimator such as those discussed in section III.A.2 to provide the missing elements of the state vector of the controllable observable canonic subsystem. This section provides a condition which is sufficient for stability of time-invariant linear regulators which meet Condition 1.

Wonham's main result of reference [W-3] is required: Let Λ be an arbitrary set of n eigenvalues. "The pair $(\underset{\sim}{A}, \underset{\sim}{B})$ is controllable if and only if, for every Λ, there is a matrix $\underset{\sim}{C}$ such that $\underset{\sim}{A} + \underset{\sim}{B}\,\underset{\sim}{C}$ has Λ for its set of eigenvalues."

The condition for stabilizability using dynamic controllers is stated and proven for perfect canonic state information and then intuitively extended to include estimated state information with exponentially decreasing error.

Condition 3. A time-invariant linear regulator system meeting Condition 1 can be stabilized with output feedback if all states of the controllable observable canonic subsystem are available.

Proof: Assume that all ℓ states of the controllable, observable system are available. Then the vector $\hat{\underset{\sim}{z}}$ which can be used for feedback is

$$\hat{\underset{\sim}{z}} = \begin{bmatrix} \underset{\sim}{0} \\ \cdots\cdots \\ \underset{\sim}{z}_2 \\ \cdots\cdots \\ \underset{\sim}{0} \\ \cdots\cdots \\ \underset{\sim}{0} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{I} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} \underset{\sim}{z} = \hat{\underset{\sim}{I}}\, \underset{\sim}{z}$$

where $\underset{\sim}{I}$ is the identity matrix of dimension ℓ. A constant gain feedback control would be

$$\underset{\sim}{u} = -\hat{\underset{\sim}{F}}\, \hat{\underset{\sim}{z}} = -\hat{\underset{\sim}{F}}\, \hat{\underset{\sim}{I}}\, \underset{\sim}{z} = -\, (\underset{\sim}{0}\ \underset{\sim}{F}\ \underset{\sim}{0}\underset{\sim}{0})\ \underset{\sim}{z}$$

where $\underset{\sim}{F}$ is an $m \times \ell$ matrix.

Using this control for the system of equation (3) yields

$$\underset{\sim}{z} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ 0 & D_{22} & 0 & D_{24} \\ 0 & 0 & D_{33} & D_{34} \\ 0 & 0 & 0 & D_{44} \end{bmatrix} \underset{\sim}{z} - \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \\ 0 \end{bmatrix} (\underset{\sim}{0}\ \underset{\sim}{F}\ \underset{\sim}{0}\underset{\sim}{0})\ \underset{\sim}{z} \quad (9)$$

$$\dot{\underset{\sim}{z}} = \begin{bmatrix} D_{11} & D_{12} - \hat{B}_1 F & D_{13} & D_{14} \\ 0 & D_{22} - \hat{B}_2 F & 0 & D_{24} \\ 0 & 0 & D_{33} & D_{34} \\ 0 & 0 & 0 & D_{44} \end{bmatrix} \underset{\sim}{z} \qquad (10)$$

Since $(\underset{\sim}{D}_{22}, \underset{\sim}{\hat{B}}_2)$ is a controllable pair, by Wonham's result a matrix $\underset{\sim}{F}$ exists that $(\underset{\sim}{D}_{22} - \underset{\sim}{\hat{B}}_2 \underset{\sim}{F})$ has any desired set of eigenvalues. Since by Condition 1 $\underset{\sim}{D}_{11}$, $\underset{\sim}{D}_{33}$ and $\underset{\sim}{D}_{44}$ have eigenvalues with negative real parts Condition 3 is proven.

The output of a state estimator, however, only approximates the state. The Luenberger observer output, for example, has an exponentially decaying error as seen in equation (10) of section VI.B. If the observer has operated a sufficiently long time for the observer output to accurately represent the state of the controllable observable subsystem, Condition 3 likewise applies.

V. TWO EXTENSIONS OF OZER'S METHOD

A. DESCRIPTION OF METHOD

A useful method of solving for the gains of an output-feedback control for a suboptimal linear regulator has been developed by Ozer [O-1]. He applied the method to the solution of constant and piecewise-constant gains for the finite-time-interval regulator. In this chapter two extensions of his method are presented: 1) a technique for finding piecewise-constant gains with variable time intervals, and 2) an algorithm for determining constant gains for infinite-time-interval problems.

The problem statement is as follows.

The system is

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}(t)\ \underset{\sim}{x} + \underset{\sim}{B}(t)\ \underset{\sim}{u}$$

$$\underset{\sim}{y} = \underset{\sim}{C}(t)\ \underset{\sim}{x}\ , \tag{1}$$

where the initial condition, $\underset{\sim}{x}(0)$, are points on a unit hypersphere.[10] The performance measure is

$$J = 1/2\underset{\sim}{x}^T(t_f)\underset{\sim}{H}\underset{\sim}{x}(t_f) + 1/2\int_{t_o}^{t_f}(\underset{\sim}{x}^T(t)\underset{\sim}{Q}(t)\underset{\sim}{x}(t) + \underset{\sim}{u}^T(t)\underset{\sim}{R}(t)\underset{\sim}{u}(t))\,dt\ ; \tag{2}$$

---

[10] It can be shown that this is equivalent to the requirement that $||x_o|| \leq \rho$ where $||\ \ ||$ denotes the Euclidian norm.

where $\underset{\sim}{H}$ and $\underset{\sim}{Q}(t)$ are real symmetric and positive semidefinite and $\underset{\sim}{R}(t)$ is real symmetric and positive definite. The output feedback control is constrained to be of the form

$$\underset{\sim}{u}_s(t) = -\underset{\sim}{P}_s(t)\underset{\sim}{y}(t) = -\underset{\sim}{P}_s(t)\underset{\sim}{C}(t)\underset{\sim}{x}(t) = -\underset{\sim}{F}_s(t)\underset{\sim}{x}(t)$$

where $\underset{\sim}{P}_s(t)$ is a member of a specified class of real time functions. The problem is to find the elements of $\underset{\sim}{P}_s(t)$ which minimize the maximum value of AD $(\underset{\sim}{x}_o, \underset{\sim}{P}_s(\cdot))$ over the unit hypersphere of initial conditions where

$$AD\ (\underset{\sim}{x}_o, \underset{\sim}{P}_s(\cdot)) = J_s - J^*$$

$$J_s = 1/2\underset{\sim}{x}_s^T(t_f)\underset{\sim}{H}\underset{\sim}{x}_s(t_f) + 1/2\int_{t_o}^{t_f}(\underset{\sim}{x}_s^T(t)\underset{\sim}{Q}(t)\underset{\sim}{x}_s(t) + \underset{\sim}{u}_s^T(t)\underset{\sim}{R}(t)\underset{\sim}{u}_s(t))dt$$

$$J^* = 1/2\underset{\sim}{x}_*^T(t_f)\underset{\sim}{H}\underset{\sim}{x}_*(t_f) + 1/2\int_{t_o}^{t_f}(\underset{\sim}{x}_*^T(t)\underset{\sim}{Q}(t)\underset{\sim}{x}_*(t) + \underset{\sim}{u}_*^T(t)\underset{\sim}{R}(t)\underset{\sim}{u}_*(t))\ dt\ .$$

A D is called the absolute degradation of performance, $\underset{\sim}{x}_s(\cdot)$ is the state trajectory resulting from application of the suboptimal control law, and $\underset{\sim}{x}_*(\cdot)$ is the optimal state trajectory.

Of interest is a matrix differential equation[11] given by

$$\dot{\underset{\sim}{V}}(t) = -\underset{\sim}{V}(t)(\underset{\sim}{A}(t) - \underset{\sim}{B}(t)\underset{\sim}{F}_s(t)) - (\underset{\sim}{A}(t) - \underset{\sim}{B}(t)\underset{\sim}{F}_s(t))^T\underset{\sim}{V}(t)$$
$$-\underset{\sim}{Q}(t) - \underset{\sim}{F}_s^T(t)\underset{\sim}{R}(t)\underset{\sim}{F}_s(t)$$

---

[11]The derivation of this equation is available in [M-2] and [K-8] and will not be given here.

whose solution must satisfy the boundary condition

$$\underset{\sim}{V}(t_f) = \underset{\sim}{H} \quad .$$

The solution $\underset{\sim}{V}(t)$, called the cost matrix by Kleinman and Athans [K-8], is a real symmetric positive definite matrix analogous to the Riccati solution in the sense that

$$J_s = 1/2 \; \underset{\sim}{x}_o^T \; \underset{\sim}{V}(t_o, \underset{\sim}{P}_s(\cdot)) \; \underset{\sim}{x}_o$$

just as

$$J^* = 1/2 \; \underset{\sim}{x}_o^T \; \underset{\sim}{K}(t_o) \; \underset{\sim}{x}_o \quad .$$

Ozer shows that minimizing the maximum absolute degradation over initial conditions on the unit hypersphere can be computationally simplified as follows:

$$\min_{\underset{\sim}{P}_s(\cdot)} \; \max_{\underset{\sim}{x}_o} AD(\underset{\sim}{x}_o, \underset{\sim}{P}_s(\cdot)) = \min_{\underset{\sim}{P}_s(\cdot)} \; \max_{\underset{\sim}{x}_o} (J(\underset{\sim}{x}_o, \underset{\sim}{P}_s(\cdot)) - J^*(\underset{\sim}{x}_o))$$

$$= \min_{\underset{\sim}{P}_s(\cdot)} \; \max_{\underset{\sim}{x}_o} \; 1/2 \; \underset{\sim}{x}_o^T \; (\underset{\sim}{V}(t_o, \underset{\sim}{P}_s(\cdot)) - \underset{\sim}{K}(t_o)) \; \underset{\sim}{x}_o$$

$$= \min_{\underset{\sim}{P}_s(\cdot)} \; \max_{\underset{\sim}{x}_o} \; 1/2 \; \underset{\sim}{x}_o^T \underset{\sim}{W}(t_o) \; \underset{\sim}{x}_o$$

$$= \min_{\underset{\sim}{P}_s(\cdot)} \; 1/2 \; \lambda_W$$

where λ_W is the value of the largest eigenvalue of $\underset{\sim}{W}(t_o)$, a real symmetric and positive definite matrix.

The steps of the method are:

1) Evaluate $\underset{\sim}{K}(t_o)$

2) Minimize with a pattern search routine[12] over the elements of $\underset{\sim}{P}_s(\cdot)$ an objective function calculated in the following manner:

a) with the current $\underset{\sim}{P}_s(\cdot)$, evaluate $\underset{\sim}{V}(t_o)$ by integration.

b) find the largest eigenvalue of $\underset{\sim}{W}(t_o) = \underset{\sim}{V}(t_o)-\underset{\sim}{K}(t_o)$.

c) assign to the objective function the value of step b).

B. PIECEWISE-CONSTANT GAINS WITH VARIABLE SWITCHING TIMES

1. Discussion

The optimal control for the linear regulator system of equation (1) which is to be optimized over a finite time interval with the quadratic performance measure of equation (2) is a feedback of all states with time-varying gains,

$$\underset{\sim}{u}_*(t) = -\underset{\sim}{R}^{-1}\underset{\sim}{B}^T(t)\underset{\sim}{K}(t)\underset{\sim}{x}(t) = -\underset{\sim}{F}^*(t)\underset{\sim}{x}(t) .$$

As previously discussed, a more easily realized control can be specified of the form

$$\underset{\sim}{u}_s(t) = -\underset{\sim}{P}_s(t)\underset{\sim}{y}(t) ,$$

where $\underset{\sim}{P}_s(t)$ is a piecewise-constant matrix. More precisely, let the time interval $[t_o,t_f]$ be divided into N equal

---

[12]See footnote, page 32.

subintervals of length Δt. During the i^{th} subinterval, $[t_{i-1}, t_{i-1} + \Delta t)$, the gain matrix is $\tilde{P}_i$. $\tilde{P}_s(t)$ is then a set of $N(m \times p)$ matrices and the resulting minimization problem has mpN variables. Because mpN can be a large number, the computation time required for solving for all gains in one problem can be lengthy; however, this computation is done off-line.

In this section the interval lengths are allowed to be variables in addition to the gains. Division of the problem into two alternating separate minimizations over mpN gains and over N-1 switching times is not likely to yield even a local minimum. This can be intuitively seen by the fact that minimizing a function of just two parameters by independent varying of each parameter does not necessarily result in a local minimum.

Thus no alternative was found other than to minimize $AD(\tilde{x}_o, \tilde{P}_s(\cdot), \tilde{t}_s)$ directly over all (mpN + N-1) parameters. Computation times for the example problem were lengthy, as expected, but the absolute degradation was significantly reduced.

The method used consisted of a pattern search in (mpN + N-1) - dimensional space to minimize the maximum absolute degradation, $J_s - J^*$. A fourth-order Runge-Kutta integration method was used. Logical statements tested that the switching times were within the interval $[t_o, t_f]$ and were correctly ordered. Prior to each integration step, the running variable (time) was tested for going past the next switching time. If

it would, the step was reduced to end the integration on the switching time. After advancing parameters and switching time values, integration continued.

The steps in the procedure are:

1. Calculate $\underset{\sim}{K}(t_o)$.

2. Guess $\underset{\sim}{p}_v$, a vector of (mpN + N-1) elements.

3. Use the pattern search library routine[13] to minimize an objective function determined as follows:

   a) Test that the switching times, t_{s_i}, i = 1,2..N-1 satisfy the inequalities: $t_o \leq t_{s_1} \leq t_{s_2} \ldots \leq t_{s_{N-1}} \leq t_f$. If not, assign a high value to the objective function and return.

   b) Integrate the cost matrix equation using the elements of $\underset{\sim}{p}_v$ to obtain $\underset{\sim}{V}(t_o, \underset{\sim}{P}_s(\cdot), \underset{\sim}{t}_s)$

   c) Find the largest eigenvalue of

   $\underset{\sim}{W}(t_o) = \underset{\sim}{V}(t_o, \underset{\sim}{P}_s(\cdot), \underset{\sim}{t}_s) - \underset{\sim}{K}(t_o)$

   d) Assign to the objective function the value of step c) and return.

[13]See footnote, page 32.

2. An Example Problem

The system[14] to be controlled is described by

$$\dot{\underset{\sim}{x}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix} \underset{\sim}{x} + \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} u$$

$$\underset{\sim}{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{x} \tag{3}$$

The performance measure is

$$J = 1/2 \int_0^2 (\underset{\sim}{x}^T(t) \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{x}(t) + u^2(t))dt \tag{4}$$

The control is constrained to be of the form

$$\underset{\sim}{u}_s(t) = -\underset{\sim}{p}_s(t)\underset{\sim}{y}(t) = -(p_1(t) \;\; p_2(t) \;\; p_3(t)) \; \underset{\sim}{x}(t) ,$$

where $\underset{\sim}{p}_s(t)$ is piecewise-constant over one, two, three or four intervals. For comparison the suboptimal gains were calculated for both fixed equal intervals and variable intervals. Table V-1 shows values of the relative degradation[15] for one, two,

---

[14] This problem has been solved by Kleinman, Fortmann and Athans in reference [K-9] and by Ozer in reference [O-1].

[15] Relative degradation is defined as the absolute degradation divided by the maximum value of J* with initial conditions on the unit hypersphere.

three and four subintervals. Optimal and suboptimal gain values for three and four intervals are shown in Figures V-1 and V-2. The values of performance measure degradation were confirmed by an independent direct integration for J_s and J^*.

RELATIVE DEGRADATION

Number of Intervals	RD of System with Fixed Switching Times	RD of System with Variable Switching Times
1	.03566	------
2	.00419	.00419
3	.00645	.00360
4	.00392	.00170

RD = Relative Degradation = Absolute Degradation / J^* max

Table V-1 Performance of the System of Example 1 with Fixed Switching Times and Variable Switching Times.

3. Discussion of Results

The results of this example show that optimizing switching times with piecewise-constant gains leads to improved performance. When switching times are variable the intervals are short when the optimal gains are changing rapidly, as would be expected. This observation holds only when all states are being fed back, for only then do the suboptimal gain curves approach the form of the optimal. The importance of matching intervals to the optimal gain curves even for fixed equal intervals is illustrated by the poorer system performance

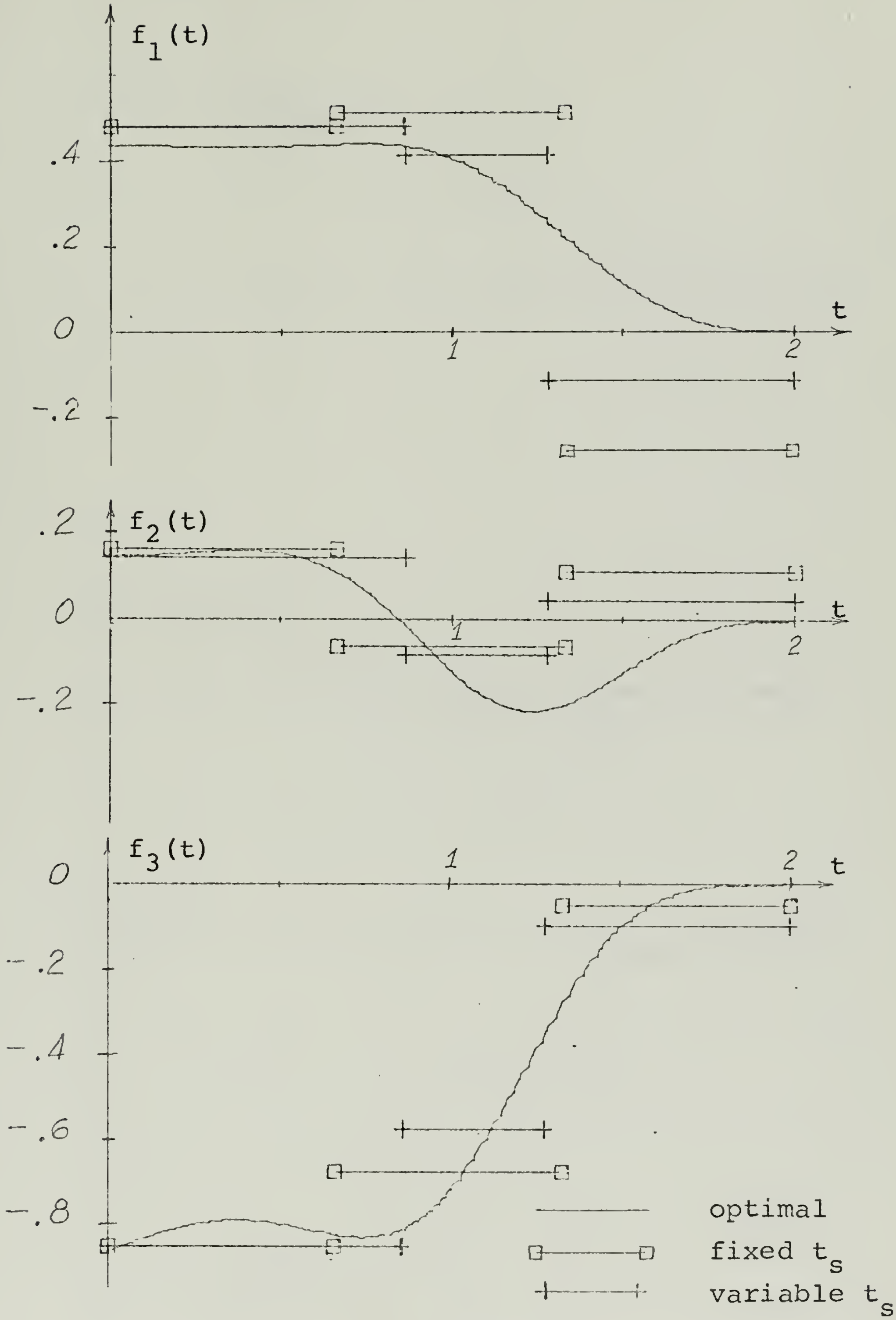


Fig. V-1. Example Problem. Three Intervals.

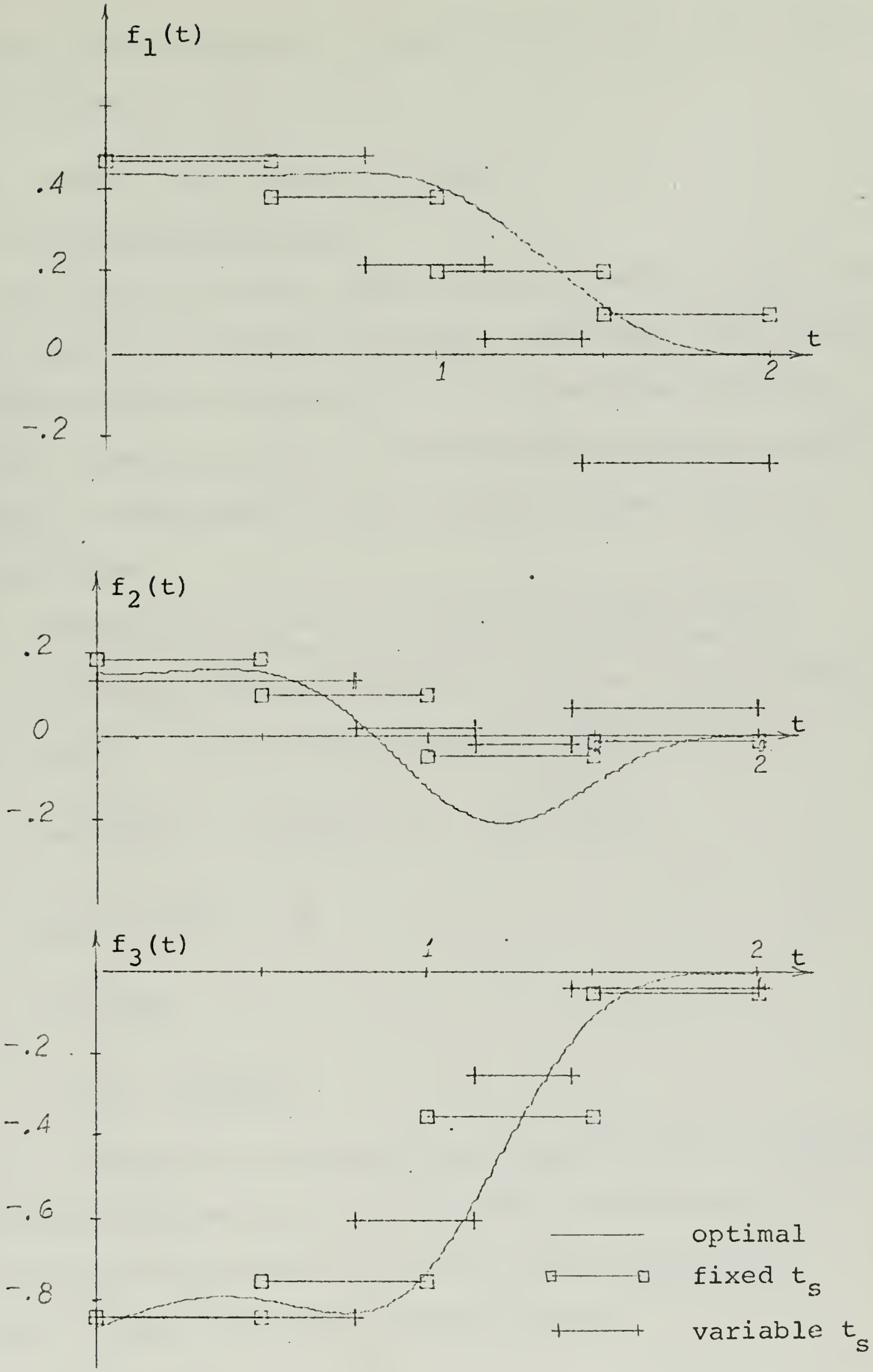


Fig. V-2. Example Problem. Four Intervals.

with three fixed intervals as compared with two. Ozer noted a similar effect.

C. THE INFINITE TIME INTERVAL PROBLEM

1. The Problem Statement

The problem is the same as that of part A except that $\underset{\sim}{A}$, $\underset{\sim}{B}$,$\underset{\sim}{C}$,$\underset{\sim}{Q}$, and $\underset{\sim}{R}$ are constant matrices, the upper limit on the performance measure integral is ∞, $\underset{\sim}{Q}$ is positive definite, and $\underset{\sim}{P}_s$ is a constant matrix. The matrix $\underset{\sim}{Q}$ is positive definite to satisfy a requirement of the solution method used, as explained below.

2. Steady-state Solution of the Cost Matrix, $\underset{\sim}{V}(t)$

The steady state solution of the cost matrix differential equation is the linear algebraic equation:

$$\underset{\sim}{V}_{ss}(\underset{\sim}{A}-\underset{\sim}{B}\underset{\sim}{F}_s) + (\underset{\sim}{A}-\underset{\sim}{B}\underset{\sim}{F}_s)^T\underset{\sim}{V}_{ss} = -\underset{\sim}{Q}-\underset{\sim}{C}^T\underset{\sim}{P}^T\underset{\sim}{R}\underset{\sim}{P}\underset{\sim}{C}$$

or $$\underset{\sim}{V}_{ss}\hat{\underset{\sim}{A}} + \hat{\underset{\sim}{A}}^T\underset{\sim}{V}_{ss} = -\hat{\underset{\sim}{Q}} \qquad (5)$$

where $$\hat{\underset{\sim}{A}} = \underset{\sim}{A}-\underset{\sim}{B}\underset{\sim}{F}_s$$

$$\hat{\underset{\sim}{Q}} = \underset{\sim}{Q} + \underset{\sim}{C}^T\underset{\sim}{P}^T\underset{\sim}{R}\underset{\sim}{P}\underset{\sim}{C}$$

Equation (5) may be recognized as an equation due to Lyapunov. The Lyapunov theorem as given by Kalman and Bertram [K-5] is:

> The equilibrium state $\underset{\sim}{x}_e = \underset{\sim}{0}$ of a continuous time free, linear, stationary dynamic system
>
> $$\dot{\underset{\sim}{x}} = (\underset{\sim}{A}-\underset{\sim}{B}\underset{\sim}{F}_s)\underset{\sim}{x} = \hat{\underset{\sim}{A}}\underset{\sim}{x} \qquad (6)$$

is asymptotically stable if and only if given any real symmetric positive definite matrix $\hat{\underset{\sim}{Q}}$ there exists a real symmetric

positive definite matrix $\underset{\sim}{V}_{ss}$ which is the unique solution of the set of n(n+1)/2 linear equations:

$$\hat{\underset{\sim}{A}}^T \underset{\sim}{V}_{ss} + \underset{\sim}{V}_{ss} \hat{\underset{\sim}{A}} = - \hat{\underset{\sim}{Q}} \tag{7}$$

and $\underset{\sim}{x}^T \underset{\sim}{V}_{ss} \underset{\sim}{x}$ is a Lyapunov function for (6). The Lyapunov theorem provides assurance that for a choice of $\hat{\underset{\sim}{A}}$ and $\hat{\underset{\sim}{Q}}$ which meet the Lyapunov theorem requirements a steady-state solution of the cost matrix can be found.

3. Solution Method

Given the system matrices, $\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{Q}$, and $\underset{\sim}{R}$, the first task is to determine that the system can be stabilized, following the methods of chapter IV. If so, the pattern search routine of IV.C.2 provides an initial $\underset{\sim}{P}_s$ which causes the eigenvalues of the closed-loop system matrix, $\hat{\underset{\sim}{A}}$, to have negative real parts. Since $\hat{\underset{\sim}{Q}} = \underset{\sim}{Q} + \underset{\sim}{P}_s^T \underset{\sim}{C}^T \underset{\sim}{R} \underset{\sim}{C} \underset{\sim}{P}_s$ is positive definite for all $\underset{\sim}{P}_s$, the assumptions for the Lyapunov theorem are met and the solution $\underset{\sim}{V}_{ss}$ of (5) can be obtained. The largest eigenvalue of $\underset{\sim}{W} = \underset{\sim}{V}_{ss} - \underset{\sim}{K}_{ss}$ is then minimized by Ozer's method.

It is essential that the procedure start with a $\underset{\sim}{P}_s$ that yields a stable system, otherwise the Lyapunov equation cannot be solved for $\underset{\sim}{V}_{ss}$.

To solve equation (5), a method of S. P. Bingulac [B-2] is used. In this technique the matrix equation (5) is changed to the form

$$\underset{\sim}{U} \underset{\sim}{v}_v = -\hat{\underset{\sim}{q}}_v \quad , \tag{8}$$

where $\underset{\sim}{v}_v$ and $\hat{\underset{\sim}{q}}_v$ are $m = n(n+1)/2$ dimensional vectors of the m distinct elements of $\underset{\sim}{V}_{ss}$ and $\hat{\underset{\sim}{Q}}$. $\underset{\sim}{U}$ is an m x m matrix constructed from the elements of $\hat{\underset{\sim}{A}}$. Equation (8) is then solved for $\underset{\sim}{v}_v$ using a standard linear equation solving routine.

4. An Example Problem

In section IV.C.3 equations were introduced describing a string of three moving vehicles with unit masses and drag coefficients. The system equations and performance measure are

$$\dot{\underset{\sim}{x}} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \underset{\sim}{x} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{u} \qquad (9)$$

$$J = \int_0^\infty (\underset{\sim}{x}^T(t) \begin{bmatrix} .1 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{x}(t) + \underset{\sim}{u}^T(t)\underset{\sim}{u}(t))\, dt \qquad (10)$$

It was confirmed that constant gain feedback of the distance deviations could stabilize the system. The form of such an output and control are shown below.

$$\underset{\sim}{y}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \underset{\sim}{x} \qquad (11)$$

$$\underset{\sim}{u}_2 = -\underset{\sim}{P}_2\underset{\sim}{y}_2 = - \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \\ p_5 & p_6 \end{bmatrix} \underset{\sim}{y}_2 \qquad (12)$$

For the system described by equations (9), (11), and (12), it is desired to determine the matrix $\underset{\sim}{P}_2$ which minimizes the cost, equation (10). This is hereafter called the "two-state feedback problem."

<u>Results for the two state feedback problem</u>.

Eight initial stable $\underset{\sim}{P}_2$ matrices found by the pattern search routine of section IV.C.3 produced local minima at four distinct points in the six dimensional space of $\underset{\sim}{P}_2$ elements. The $\underset{\sim}{P}_2$ matrices, the associated closed-loop eigenvalues and the maximum and minimum degradations are shown in Table V-2. The $\underset{\sim}{P}_2$ matrices, though distinct, are in a neighborhood, suggesting that further computational perserverance would lead to their convergence. The resulting eigenvalues are also close. Note that the eigenvalue $\lambda = -1$ of the controllable, but not observable canonic state is unchanged.

Results of the Two-State Feedback Problem

	$\underset{\sim}{P}_2$	Closed-loop eigenvalues
#1	$\begin{bmatrix} 1.81 & 0.49 \\ -1.10 & 1.54 \\ -0.50 & -1.36 \end{bmatrix}$	$-0.50 \pm j\,1.92$ $-0.50 \pm j\,1.27$ $-1.$
#2	$\begin{bmatrix} 1.23 & 0. \\ -1.39 & 1.25 \\ -0.26 & -1.62 \end{bmatrix}$	$-0.50 \pm j\,1.92$ $-0.50 \pm j\,1.14$ $-1.$
#3	$\begin{bmatrix} 1.58 & 0.27 \\ -1.39 & 1.25 \\ -0.41 & -1.73 \end{bmatrix}$	$-0.50 \pm j\,1.92$ $-0.50 \pm j\,1.37$ $-1.$
#4	$\begin{bmatrix} 1.27 & -0.05 \\ -1.59 & 1.04 \\ -0.49 & -1.81 \end{bmatrix}$	$-0.50 \pm j\,1.92$ $-0.50 \pm j\,1.23$ $-1.$

All yielded:

	AD	RD
max	1.91	0.37
min	0.04	0.008

AD = absolute degradation

RD = relative degradation = A D/J* max

TABLE V-2

If, in addition, the velocity deviation of the first vehicle is measured and used for feedback control, better performance of the system could be expected. This will be a second example.

$$\underset{\sim}{y}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \underset{\sim}{x} \tag{13}$$

$$\underset{\sim}{u}_3 = -\underset{\sim}{P}_3 \underset{\sim}{y}_3 = - \begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{bmatrix} \underset{\sim}{y}_3 \tag{14}$$

For the system described by equations (9), (13), and (14), it is desired to determine the matrix $\underset{\sim}{P}_3$ which minimizes the cost, equation (10). This is the "three-state feedback problem."

Results for the three-state feedback problem.

Three initial stable $\underset{\sim}{P}_3$ matrices each produced a distinct point in the nine-dimensional space of $\underset{\sim}{P}_3$ elements. The resulting system complex eigenvalues are grouped, but the real eigenvalue varies widely, as shown in Table V-3 and Figure V-7.

The performance of the system with two and three-state feedback is shown on Figures V-3 through V-6. The "worst" performance occurs for the set of initial conditions associated with the largest eigenvalue, λ_W, of $\underset{\sim}{W}(t_o)$, that is, the point on the surface of the unit hypersphere colinear

with the eigenvector of λ_W. All other initial conditions result in performance closer to the optimal, in the sense that J_s is closer to J^*. For the initial conditions associated with the smallest eigenvector, system performance is nearest optimal.

Results of the Three-State Feedback Problem

	$\underset{\sim}{P}_3$			ADmax / ADmin	RDmax / RDmin	Closed-loop eigenvalues
#1	5.18	4.16	0.66	1.61	0.31	$-0.57 \pm j\ 1.83$
	-0.45	-1.50	1.31	0.08	.02	$-0.89 \pm j\ 0.79$
	-0.51	-0.50	-1.69			-5.26
#2	2.06	2.46	0.005	1.57	0.30	$-0.62 \pm j\ 1.84$
	-0.63	-1.21	1.29	0.01	.002	$-1.22 \pm j\ 0.69$
	-0.25	-0.07	-1.75			-1.37
#3	3.94	3.40	0.42	1.59	0.31	$-0.58 \pm j\ 1.84$
	-1.39	-2.	1.25	0.02	.004	$-1.01 \pm j\ 0.52$
	-1.64	-1.	-1.75			-3.76

TABLE V-3

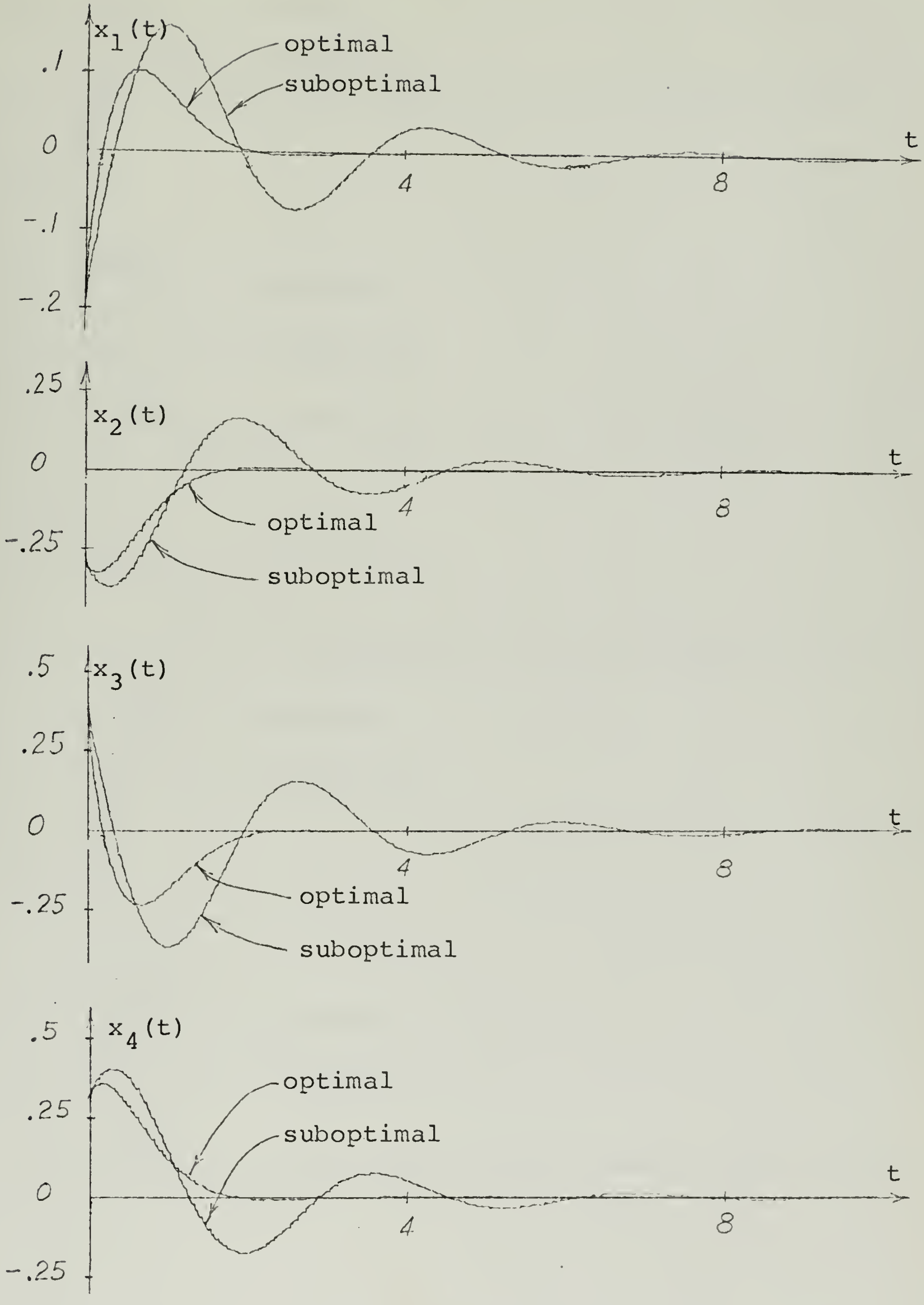


Fig. V-3. Two-state feedback trajectories. Worst performance.

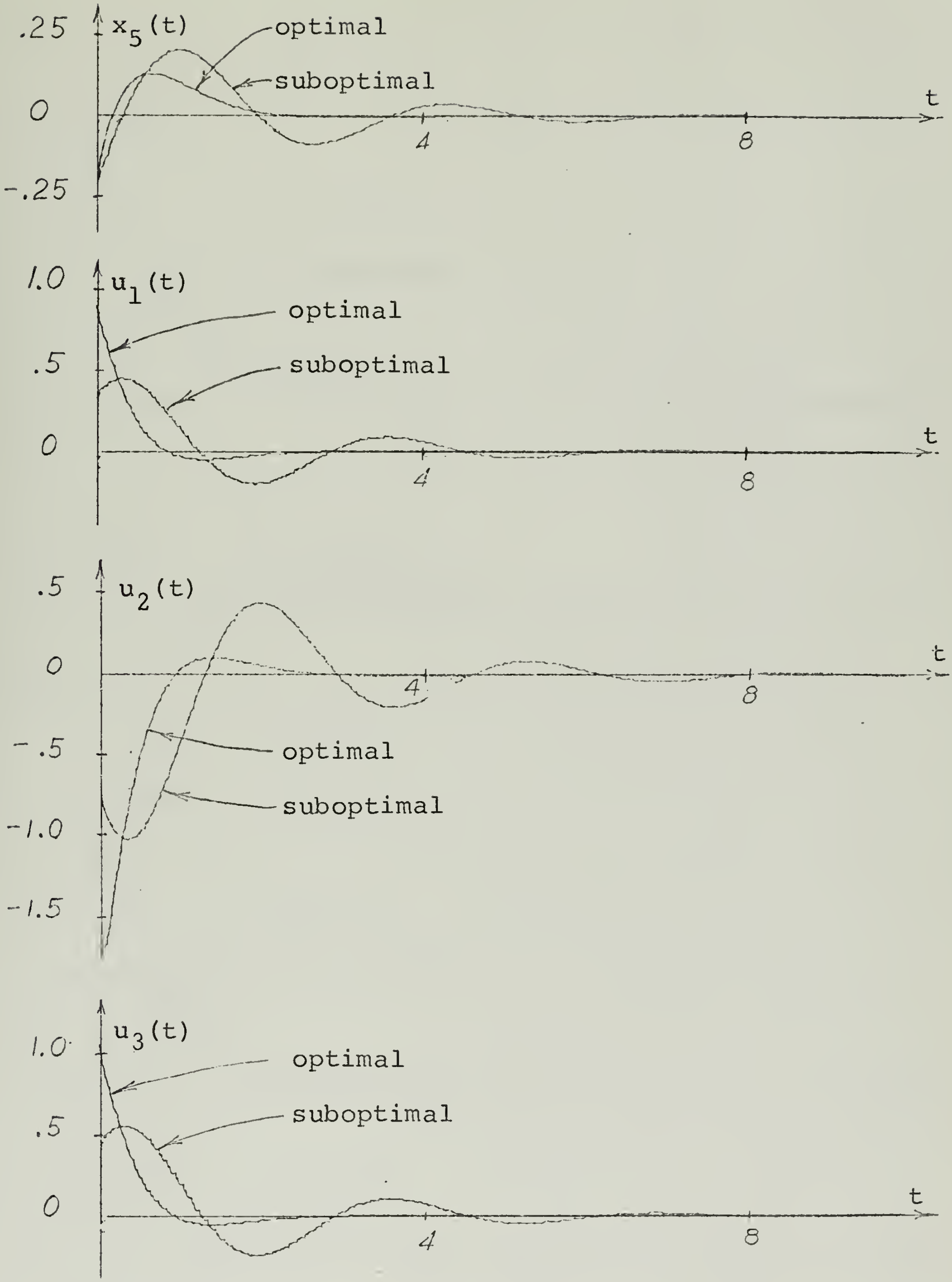


Fig. V-3. Continued.

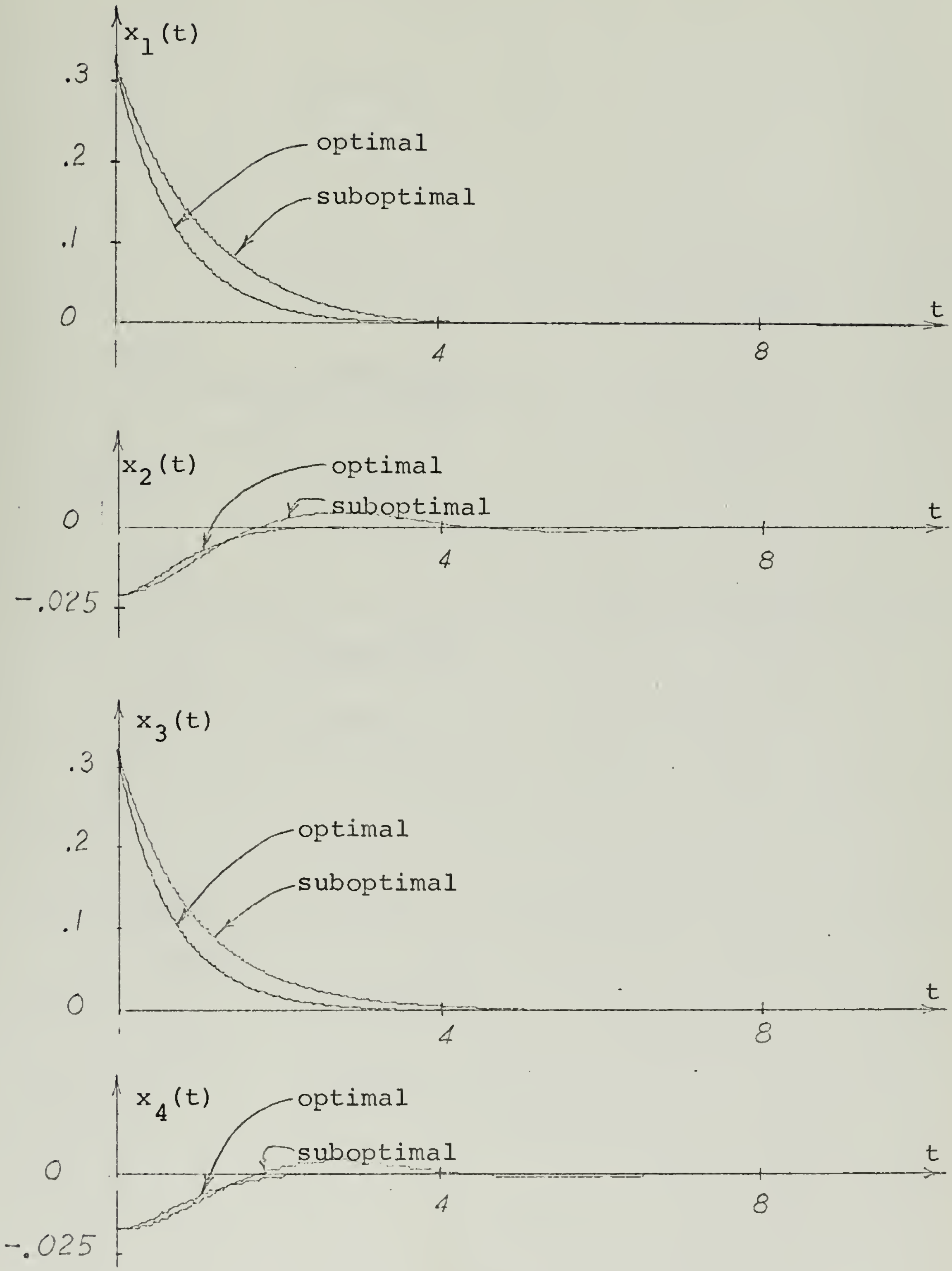


Fig. V-4. Two-state feedback trajectories. Best performance.

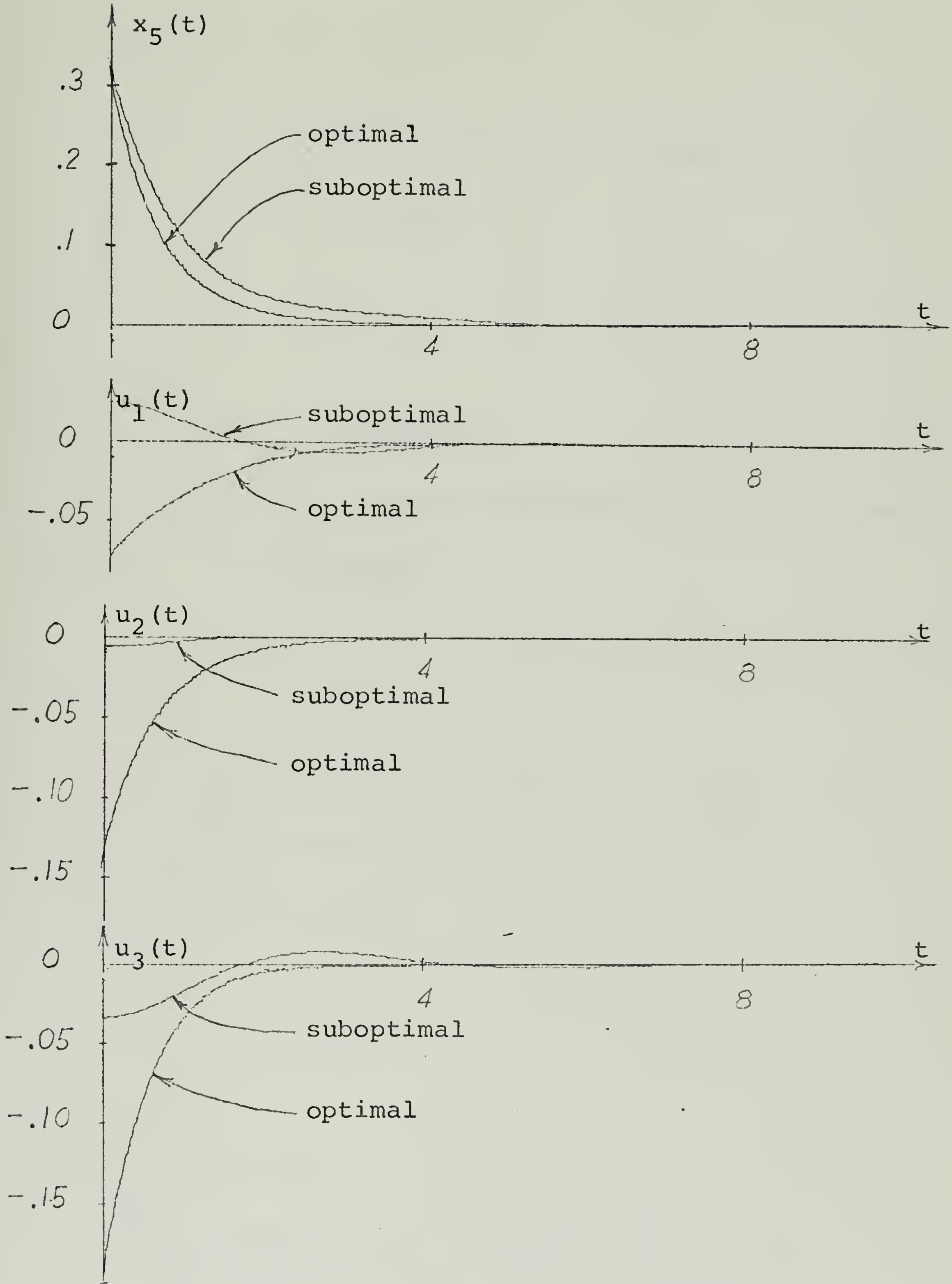


Fig. V-4. Continued.

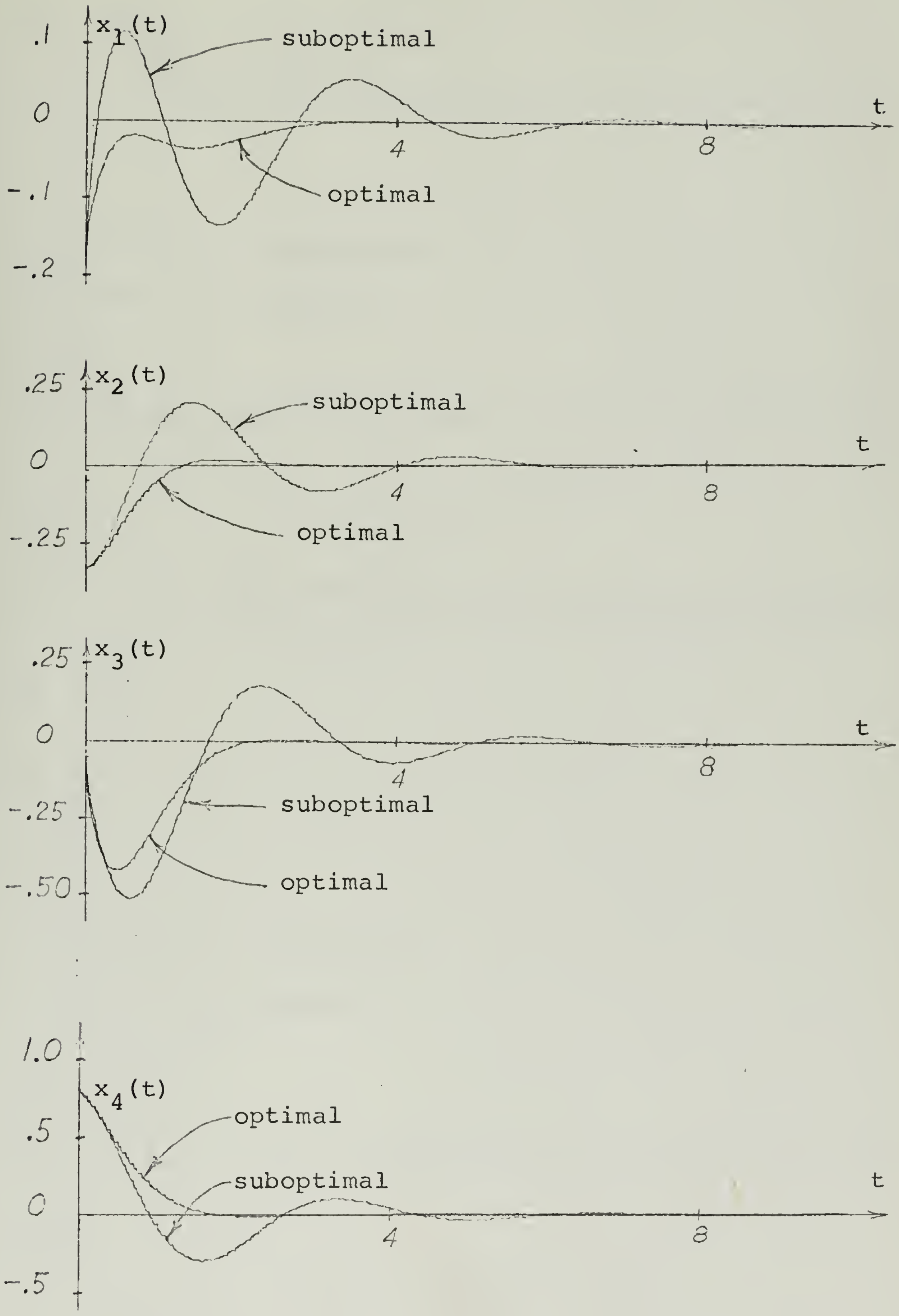


Fig. V-5. Three-state feedback trajectories. Worst performance.

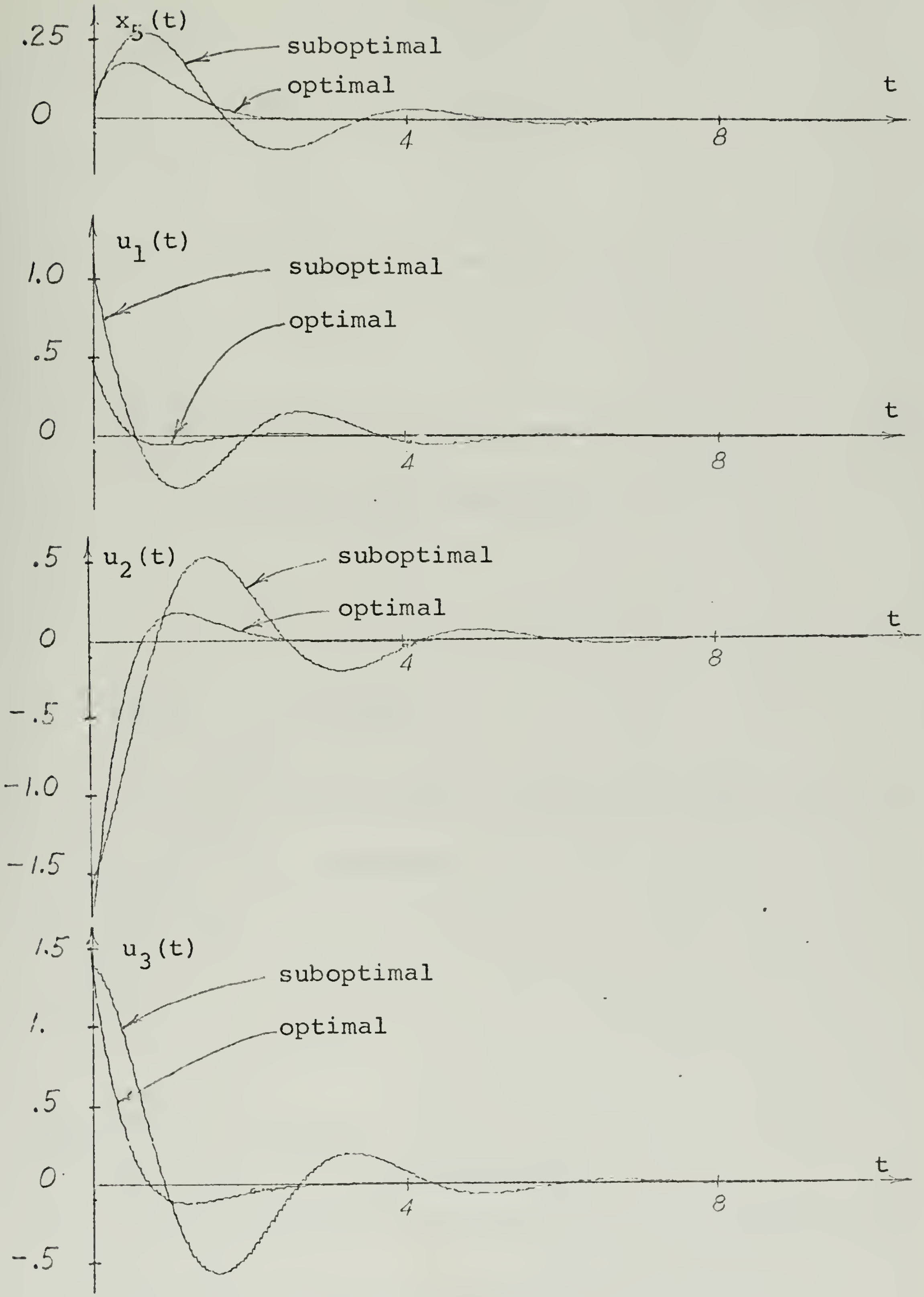


Fig. V-5. Continued.

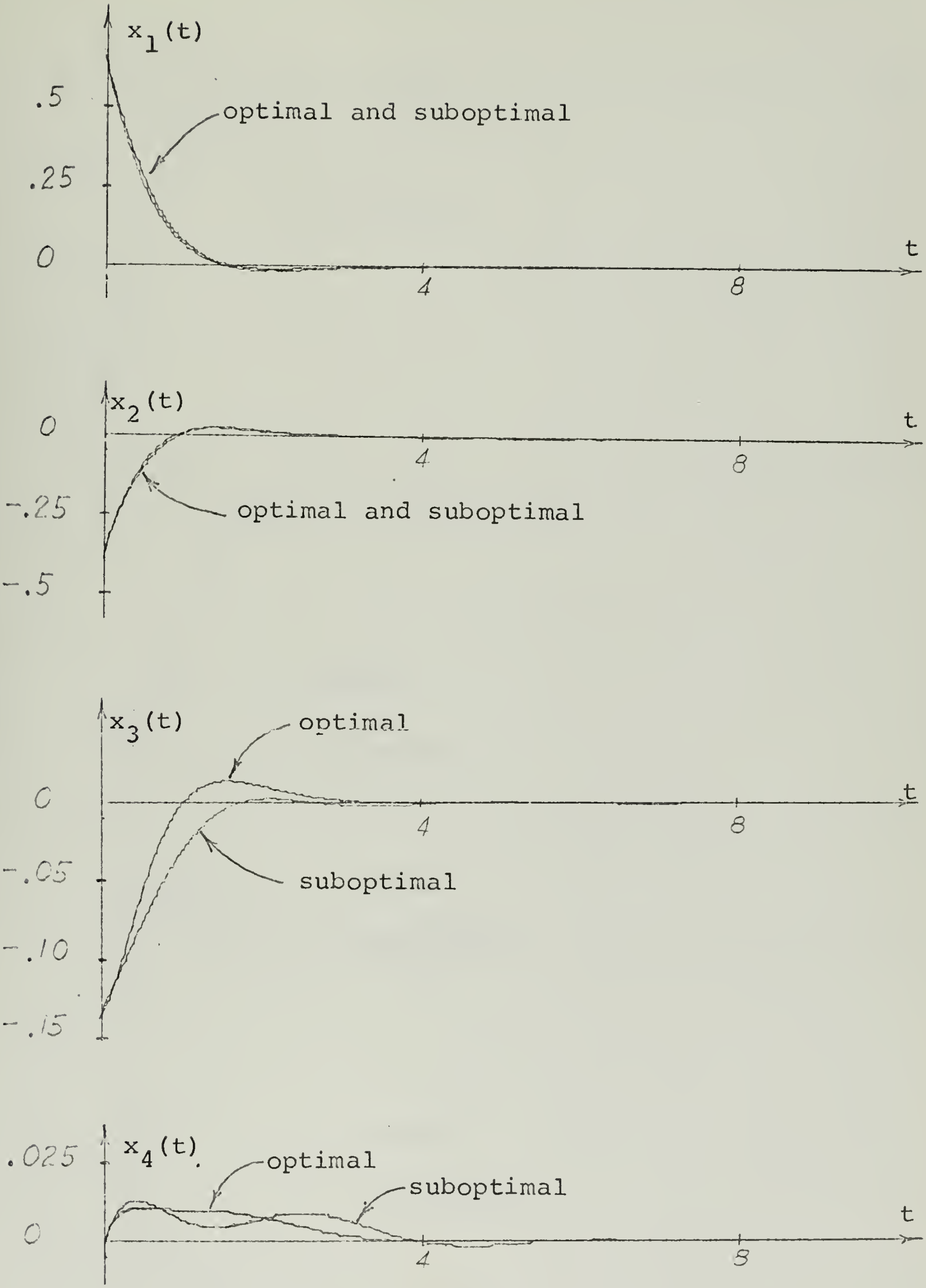


Fig. V-6. Three-state feedback trajectories.
Best performance.

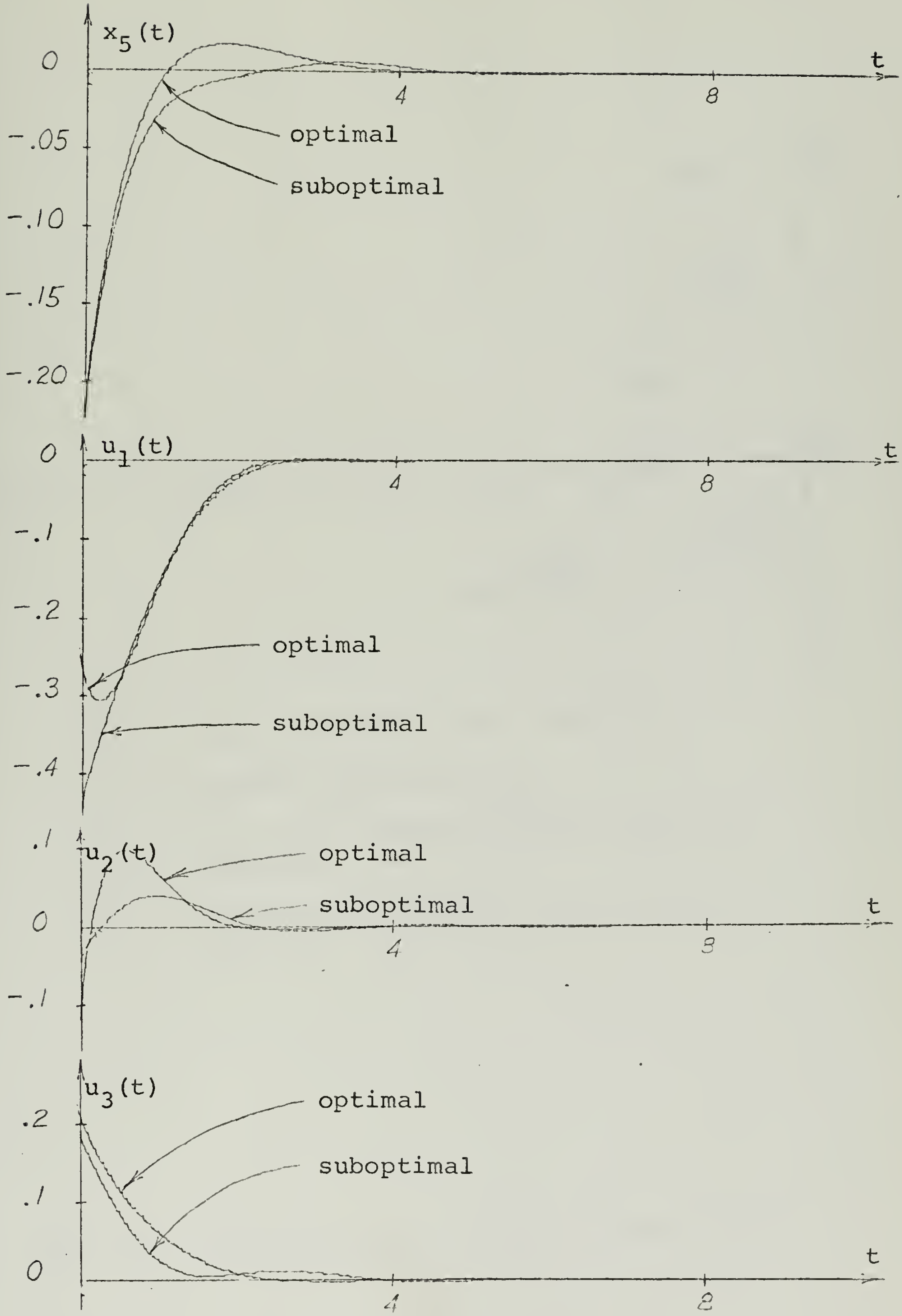
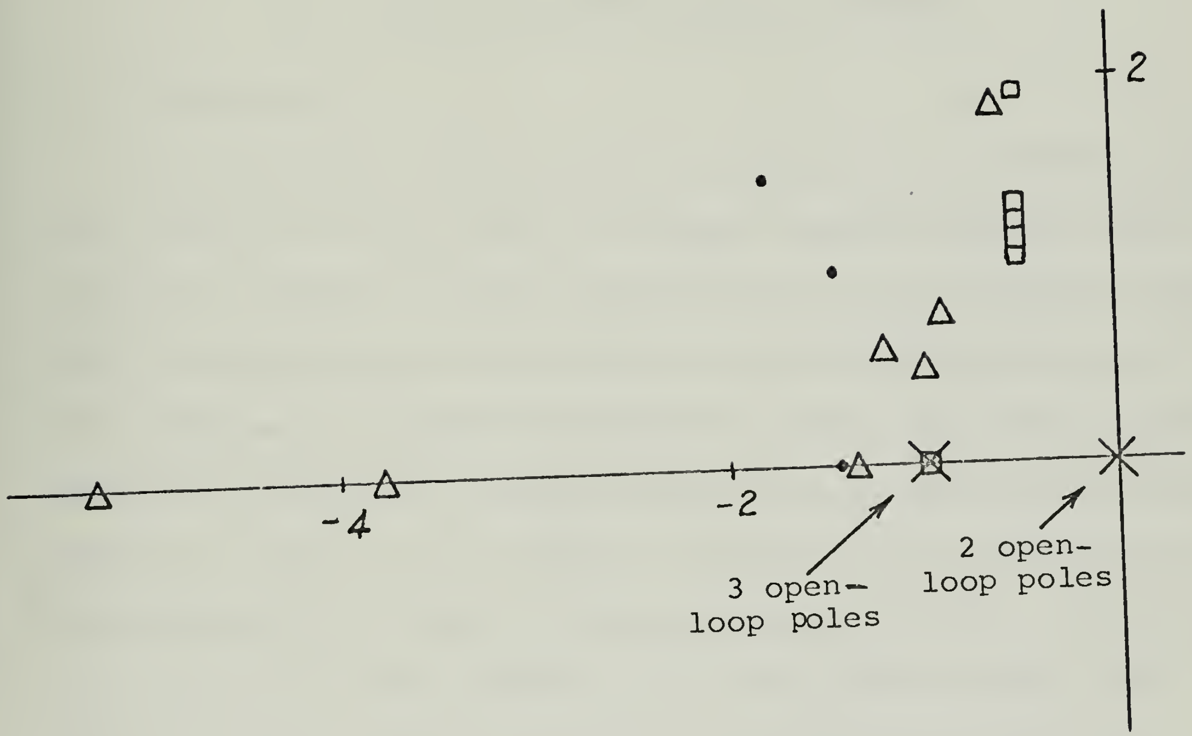


Fig. V-6. Continued.



X --- Plant open-loop pole location.

• --- Optimal pole location.

□ --- Two state feedback closed-loop pole location, four results.

△ --- Three state feedback closed-loop pole location, three results.

Fig. V-7. Poles of Moving Vehicle String System.

VI. PARTIAL CANONIC STATE FEEDBACK

A. INTRODUCTION

From information in previous chapters, it is evident that for a time-invariant linear regulator system output feedback can have no effect on the eigenvalues of the subsystems other than the completely controllable completely observable one. The performance degradation caused by those eigenvalues which cannot be influenced clearly depends on the particular system matrices; nothing can be done to improve their location without modifying the plant or the measurement method. With such a limitation on the "goodness" of system performance, the designer should exploit fully the controllable observable subsystem.

Wonham, in reference [W-3], shows that the system

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\,\underset{\sim}{x} + \underset{\sim}{B}\,\underset{\sim}{u}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{x}$$

is completely controllable if and only if for every set of arbitrary pole locations a state feedback matrix $\underset{\sim}{F}$ exists such that $(\underset{\sim}{A} + \underset{\sim}{B}\,\underset{\sim}{F})$ has the desired eigenvalues. For constant-gain output feedback the form of $\underset{\sim}{F}$ is constrained, $\underset{\sim}{F} = \underset{\sim}{P}\,\underset{\sim}{C}$. The closed-loop poles can then only be located at positions where a $\underset{\sim}{P}$ exists such that $\underset{\sim}{P}\,\underset{\sim}{C} = \underset{\sim}{F}_R$, $\underset{\sim}{F}_R$ being the required gain matrix. It would be expected that if all of the canonic states of the completely controllable completely observable

subsystem were accessible, overall system performance with a specified controller structure would be enhanced. This chapter proposes a suboptimal control method which incorporates a Luenberger state estimator for the missing elements of the completely controllable, completely observable canonic state vector, $\underset{\sim}{z}_2$. The inverse canonic transformation yields what is called "partial canonic state feedback." As an example the control of a moving vehicle string is considered. A second-order dynamic controller for it provides near optimal performance with only two states available.

B. DISCUSSION

The equations of a linear, time-invariant system in canonic form are

$$\begin{bmatrix} \dot{\underset{\sim}{z}}_1 \\ \dot{\underset{\sim}{z}}_2 \\ \dot{\underset{\sim}{z}}_3 \\ \dot{\underset{\sim}{z}}_4 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ 0 & D_{22} & 0 & D_{24} \\ 0 & 0 & D_{33} & D_{34} \\ 0 & 0 & 0 & D_{44} \end{bmatrix} \begin{bmatrix} \underset{\sim}{z}_1 \\ \underset{\sim}{z}_2 \\ \underset{\sim}{z}_3 \\ \underset{\sim}{z}_4 \end{bmatrix} + \begin{bmatrix} \hat{\underset{\sim}{B}}_1 \\ \hat{\underset{\sim}{B}}_2 \\ \underset{\sim}{0} \\ \underset{\sim}{0} \end{bmatrix} \underset{\sim}{u}$$

$$\underset{\sim}{y} = (\underset{\sim}{0} \;\; \hat{\underset{\sim}{C}}_1 \;\; \underset{\sim}{0} \;\; \hat{\underset{\sim}{C}}_2) \begin{bmatrix} \underset{\sim}{z}_1 \\ \underset{\sim}{z}_2 \\ \underset{\sim}{z}_3 \\ \underset{\sim}{z}_4 \end{bmatrix} . \qquad (1)$$

The dynamics of the completely observable subsystems are described by

$$\begin{bmatrix} \dot{z}_2 \\ \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} \dot{z}_{ob} \end{bmatrix} = \begin{bmatrix} D_{22} & D_{24} \\ \\ 0 & D_{44} \end{bmatrix} \begin{bmatrix} z_2 \\ \\ z_4 \end{bmatrix} + \begin{bmatrix} \hat{B}_2 \\ \\ 0 \end{bmatrix} u$$

or $$\dot{z}_{ob} = D_o \, z_{ob} + \begin{bmatrix} \hat{B}_2 \\ \\ 0 \end{bmatrix} u \qquad (2)$$

$$y = (\hat{C}_1 \; \hat{C}_2) \; z_{ob} = \hat{C} \; z_{ob} \qquad (3)$$

Following Luenberger [L-2], the designer looks for an auxiliary dynamic system, called an observer, producing a linear transformation L of z_{ob}. L must be such that the matrix $\begin{bmatrix} L \\ \hat{C} \end{bmatrix}$ is invertible so that $\begin{bmatrix} w \\ \\ y \end{bmatrix} = \begin{bmatrix} L \\ \\ \hat{C} \end{bmatrix} \begin{bmatrix} z_{ob} \end{bmatrix}$ may be solved for z_{ob}.

The vector $w = L \, z_{ob}$ is of order n-p where n is the order of the completely observable system and p is the number of independent outputs.

The observer system equation is

$$\dot{w} = Mw + \hat{C} \; z_{ob} + L \begin{bmatrix} \hat{B}_2 \\ \\ 0 \end{bmatrix} u \qquad (4)$$

while the equations for the free system and its observer are

$$\dot{z}_{ob} = D_o \; z_{ob} \qquad (5)$$

$$\dot{w} = Mw + \hat{C} \; z_{ob} \qquad (6)$$

where M is to be chosen.

The input to the free observer is the output $\underset{\sim}{y}$ of the system. Multiplying equation (5) by $\underset{\sim}{L}$ and substituting $\underset{\sim}{w} = \underset{\sim}{L}\, \underset{\sim}{z}_{ob}$ in equation (6) yields

$$\underset{\sim}{L}\, \dot{\underset{\sim}{z}}_{ob} = \underset{\sim}{L}\, \underset{\sim}{D}_o\, \underset{\sim}{z}_{ob} \tag{7}$$

$$\underset{\sim}{L}\, \dot{\underset{\sim}{z}}_{ob} = \underset{\sim}{M}\, \underset{\sim}{L}\, \underset{\sim}{z}_{ob} + \hat{\underset{\sim}{C}}\, \underset{\sim}{z}_{ob} \quad . \tag{8}$$

Thus the following equation must be satisfied so that $\underset{\sim}{w} = \underset{\sim}{L}\, \underset{\sim}{z}_{ob}$:

$$\underset{\sim}{L}\, \underset{\sim}{D}_o = \underset{\sim}{M}\, \underset{\sim}{L} + \hat{\underset{\sim}{C}} \quad . \tag{9}$$

Equation (9) can be solved for $\underset{\sim}{L}$ if the eigenvalues of $\underset{\sim}{M}$ are distinct from the eigenvalues of $\underset{\sim}{D}_o$. The solution of equation (6),

$$\underset{\sim}{w}(t) = \underset{\sim}{L}\, \underset{\sim}{z}_{ob}(t) + \varepsilon^{\underset{\sim}{M}t}\, (\underset{\sim}{w}(0) - \underset{\sim}{L}\, \underset{\sim}{z}_{ob}(0)), \tag{10}$$

is the estimate of $\underset{\sim}{L}\, \underset{\sim}{z}_{ob}(t)$. It can be seen that the proper choice of $\underset{\sim}{w}(0) = \underset{\sim}{L}\, \underset{\sim}{z}_{ob}(0)$ causes the observer output to be correct for all t. Since this choice of $\underset{\sim}{w}(0)$ is usually not possible, the eigenvalues of $\underset{\sim}{M}$ are chosen such that the transient error due to incorrect initial conditions will die out quickly. From $\underset{\sim}{w}(t)$ and $\underset{\sim}{y}(t)$ the elements of the observable canonic state vector can be obtained by suitable multiplications and additions.

Finally, the estimate of the controllable observable portion of the state vector, defined as $\hat{\underset{\sim}{x}}(t)$, is recovered by the canonic transformation $\underset{\sim}{T}$,

$$\hat{\underset{\sim}{x}} = \underset{\sim}{T} \begin{bmatrix} \underset{\sim}{0} \\ \cdots\cdots \\ \underset{\sim}{z}_2 \\ \cdots\cdots \\ \underset{\sim}{0} \\ \underset{\sim}{0} \end{bmatrix} \tag{11}$$

A block diagram is shown in Figure VI-1.

The information content of this partial state vector is dependent on the system. It contains, however, all of the state information which is available. In the example following, the use of partial canonic state vector feedback results in system performance superior to constant-gain output feedback of the same states.

C. AN EXAMPLE

Repeated below is equation (8) of section IV-C.3, the canonic form for the system of a string of three moving vehicles with only the two independent distance deviations accessible.

$$\begin{bmatrix} \dot{z}_1 \\ \hline \\ \\ \dot{\underset{\sim}{z}}_2 \\ \\ \end{bmatrix} = \left[\begin{array}{c|cccc} -1 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} z_1 \\ \hline \\ \\ \underset{\sim}{z}_2 \\ \\ \end{bmatrix} + \left[\begin{array}{ccc} 0 & 0 & 1 \\ \hline 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \underset{\sim}{u} \tag{12}$$

$$\underset{\sim}{y} = \left[\begin{array}{c|cccc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} z_1 \\ \underset{\sim}{z}_2 \end{bmatrix} \tag{13}$$

Here $\underset{\sim}{z}_4$ is zero-dimensional. Thus $\underset{\sim}{z}_{ob} = \underset{\sim}{z}_2$.

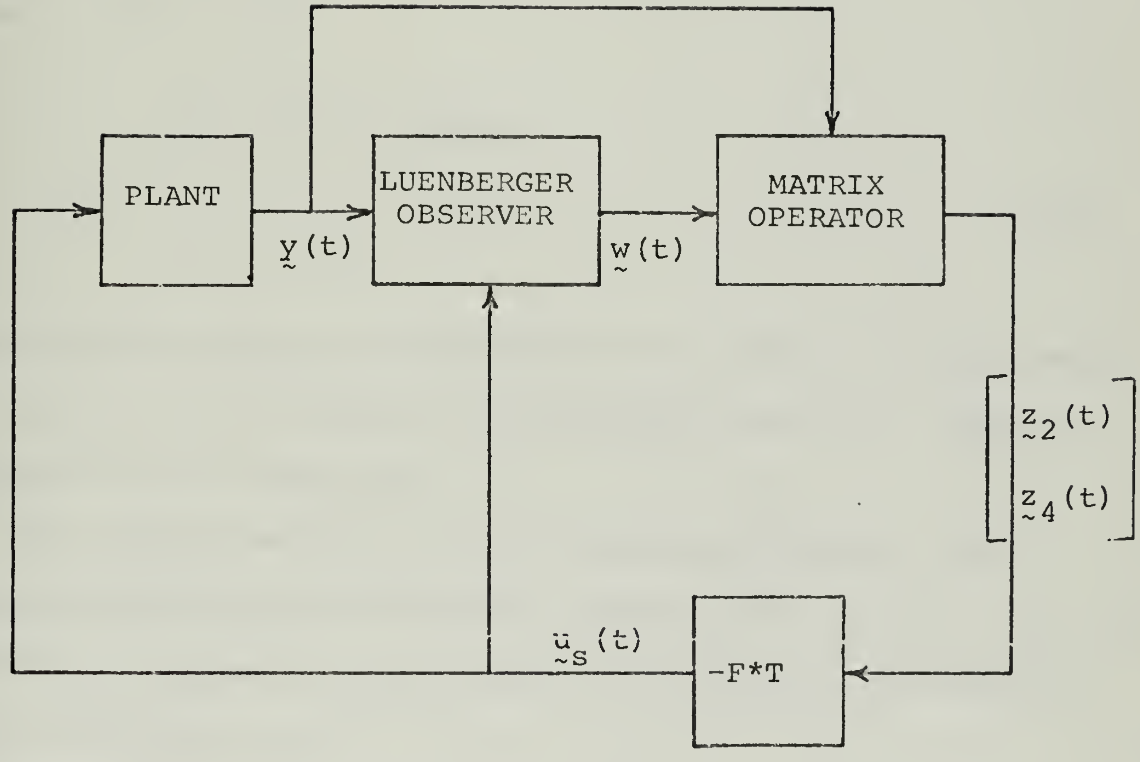


Fig VI-1. Block Diagram of Partial Canonic State Feedback System.

The observer system

$$\dot{\underset{\sim}{w}} = \underset{\sim}{M}\,\underset{\sim}{w} + \hat{\underset{\sim}{C}}_1\,\underset{\sim}{z}_2 + \underset{\sim}{L}\,\hat{\underset{\sim}{B}}_2\,\underset{\sim}{u} \tag{14}$$

is to provide (in combination with $\underset{\sim}{y}$) an estimate of the state of the system

$$\begin{aligned} \dot{\underset{\sim}{z}}_2 &= \underset{\sim}{D}_{22}\,\underset{\sim}{z}_2 + \hat{\underset{\sim}{B}}_2\,\underset{\sim}{u} \\ \underset{\sim}{y} &= \hat{\underset{\sim}{C}}_1\,\underset{\sim}{z}_2 \ . \end{aligned} \tag{15}$$

The observer system is of the order n-p, where n is the order of $\underset{\sim}{z}_2$ and p is the number of independent outputs. Thus the observer is of order two.

The transformation $\underset{\sim}{L}$ is calculated as follows. The eigenvalues of $\underset{\sim}{M}$ are arbitrarily chosen to be -3 and -4 in order to have $\underset{\sim}{w}(t)$ approach $\underset{\sim}{L}\,\underset{\sim}{z}_2(t)$ quickly,[16] but with satisfactory noise rejection [A-1].

Then

$$\underset{\sim}{L}\,\underset{\sim}{D}_{22} - \underset{\sim}{M}\,\underset{\sim}{L} = \hat{\underset{\sim}{C}}_1 \tag{16}$$

gives

$$\underset{\sim}{L}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}\underset{\sim}{L} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

---

[16] See equation (10).

$$\begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} & \ell_{14} \\ \ell_{21} & \ell_{22} & \ell_{23} & \ell_{24} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} -$$

$$\begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} & \ell_{14} \\ \ell_{21} & \ell_{22} & \ell_{23} & \ell_{24} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solving for $\underset{\sim}{L}$ yields

$$\underset{\sim}{L} = \begin{bmatrix} -1/6 & 1/6 & 1/3 & 0 \\ 0 & -1/2 & 0 & 1/4 \end{bmatrix}. \tag{17}$$

The equations of the observer system are then

$$\dot{\underset{\sim}{w}} = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} \underset{\sim}{w} + \underset{\sim}{y} + \begin{bmatrix} -1/6 & 1/6 & 0 \\ 0 & -1/12 & 1/12 \end{bmatrix} \underset{\sim}{u} \,. \tag{18}$$

The vector $\underset{\sim}{z}_2$ is reconstructed from $\underset{\sim}{w}$ and $\underset{\sim}{y}$ as follows:

$$\underset{\sim}{y} = \hat{\underset{\sim}{C}}_1 \, \underset{\sim}{z}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_{12} \\ z_{22} \\ z_{32} \\ z_{42} \end{bmatrix} = \begin{bmatrix} z_{32} \\ z_{42} \end{bmatrix} \tag{19}$$

$$\underset{\sim}{w} = \underset{\sim}{L} \, \underset{\sim}{z}_2 = \underset{\sim}{L} \begin{bmatrix} z_{12} \\ z_{22} \\ z_{32} \\ z_{42} \end{bmatrix} = 1/6 \begin{bmatrix} -z_{12} + z_{22} + 2z_{32} \\ -1/2z_{22}+3/2z_{42} \end{bmatrix}$$

$$= 1/6 \begin{bmatrix} -z_{12} + z_{22} + 2y_1 \\ -1/2z_{22}+3/2y_2 \end{bmatrix} \tag{20}$$

Solving equation (20) for z_{12} and z_{22} and combining with equation (19) yields

$$\underset{\sim}{z}_2 = \begin{bmatrix} 2 & 3 \\ 0 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \underset{\sim}{y} + \begin{bmatrix} -6 & -12 \\ 0 & -12 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \underset{\sim}{w}. \tag{21}$$

The state vector $\underset{\sim}{x}(t)$ is related to $\underset{\sim}{z}(t)$ by the canonic transformation calculated in section IV.C.3,

$$\underset{\sim}{x}(t) = \underset{\sim}{T}\, \underset{\sim}{z}(t) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{\sim}{z}(t) . \tag{22}$$

Defining $\hat{\underset{\sim}{x}}(t)$ to be the observable portion of $\underset{\sim}{x}(t)$, it is seen that

$$\hat{\underset{\sim}{x}}(t) = \underset{\sim}{T} \begin{bmatrix} 0 \\ \cdots\cdots \\ \underset{\sim}{z}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\sim}{z}_2(t) \tag{23}$$

where the column of $\underset{\sim}{T}$ associated with the unobservable canonic state has been deleted.

$$\text{Finally} \quad \hat{\underset{\sim}{x}}(t) = \begin{bmatrix} z_{12}(t) \\ z_{32}(t) \\ z_{22}(t) \\ z_{42}(t) \\ 0 \end{bmatrix} . \tag{24}$$

Since two of the elements of $\underset{\sim}{z}_2(0)$ comprise $\underset{\sim}{y}(0)$, the initial conditions for the observer system can be approximated with $\underset{\sim}{y}(0)$:

$$\underset{\sim}{w}(0) = \underset{\sim}{L}\ \underset{\sim}{z}_2(0) \cong \underset{\sim}{L} \begin{bmatrix} 0 \\ 0 \\ y_1(0) \\ y_2(0) \end{bmatrix} \cong \begin{bmatrix} 1/3y_1(0) \\ 1/4y_2(0) \end{bmatrix} \tag{25}$$

Equations (18), (21), (22) and (25) define the partial canonic state estimator. The controller is completed by using optimal gains.

$$\begin{aligned} \underset{\sim}{u}_s(t) &= -\underset{\sim}{R}^{-1}\ \underset{\sim}{B}^T\ \underset{\sim}{K}_{ss}\ \hat{\underset{\sim}{x}}(t) \\ &= -\underset{\sim}{F} * \hat{\underset{\sim}{x}}(t) \end{aligned}$$

Trajectories of the vehicle string using this controller and the optimal controller are shown in Figure VI-2 and VI-3. Initial conditions are those found to be worst for the two state and three state feedback cases of the previous chapter.

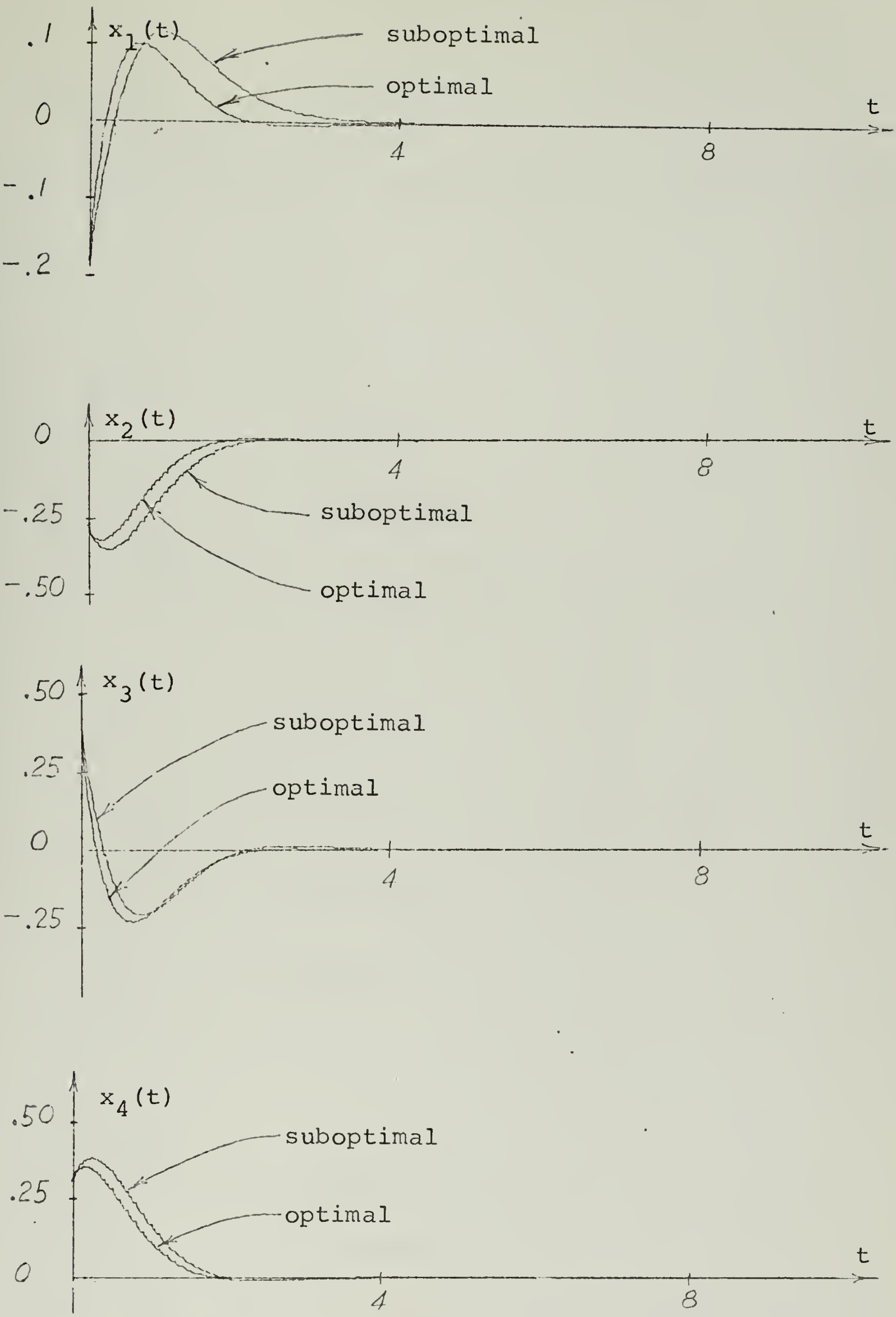


Fig. VI-2. Trajectories for system with partial canonic state feedback. Initial conditions of worst two-state feedback performance.

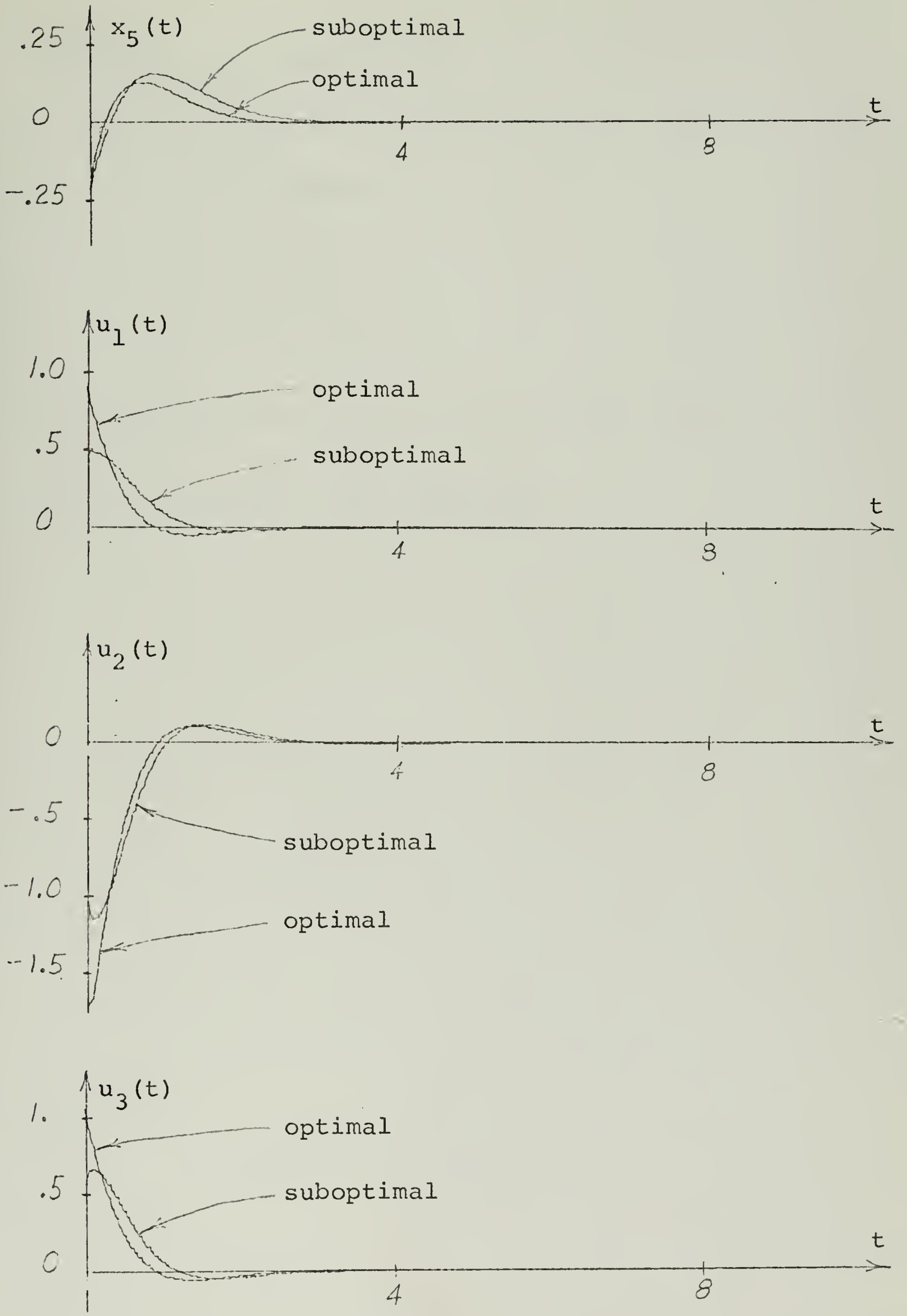


Fig. VI-2. Continued.

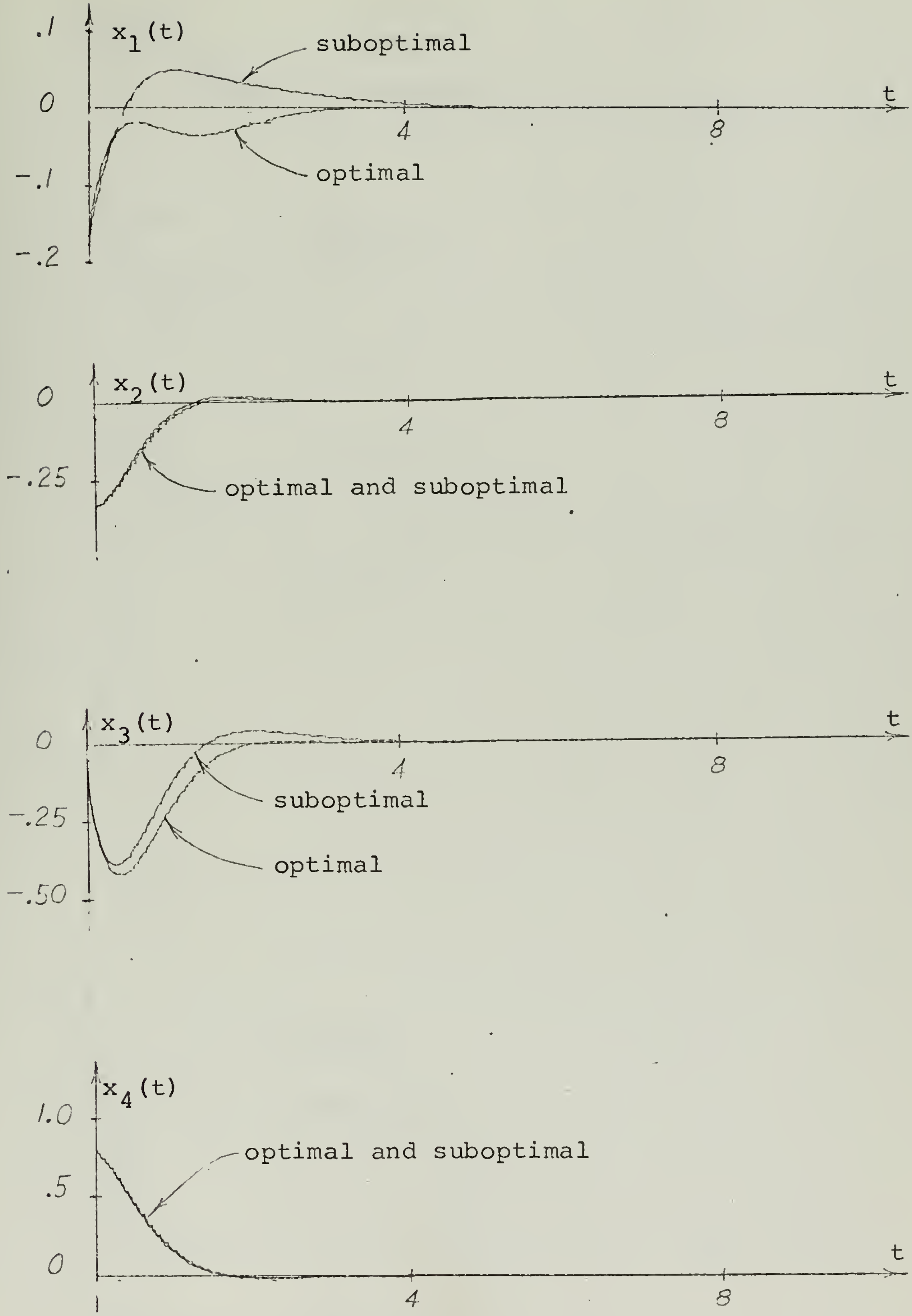


Fig. VI-3. Trajectories for system with partial canonic state feedback. Initial conditions of worst three-state feedback performance.

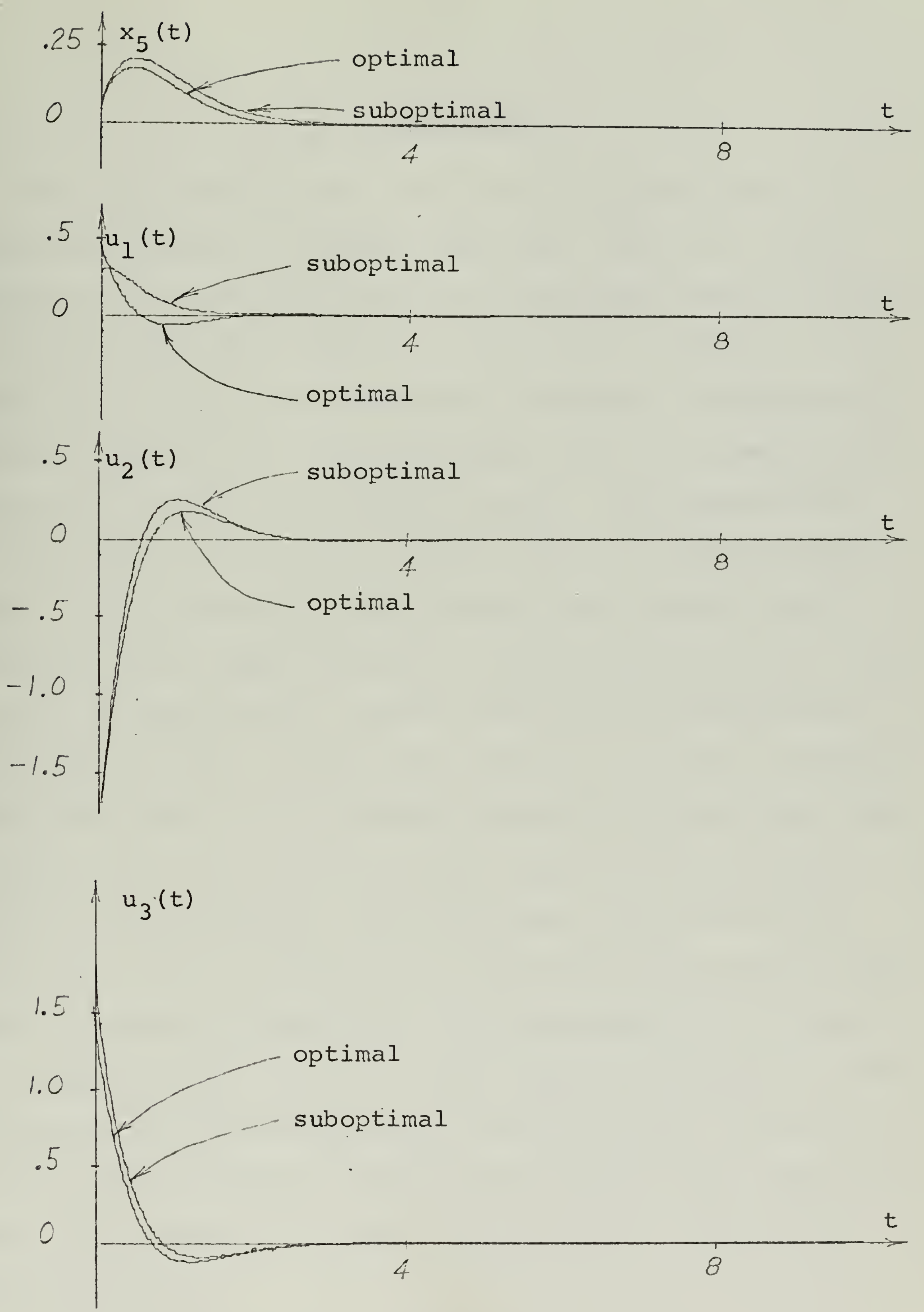


Fig. VI-3. Continued.

VII. CONCLUSIONS

The objective in this effort was the investigation of suboptimal control of linear regulators. With the Kalman decomposition, the time-invariant system was readily analyzed for stabilizability, both for constant gain feedback and for controllers which contain dynamic estimators. Combining the canonic system with a state estimator led to a promising non-iterative method of suboptimal control. The method needs further development; only the basic idea has been presented here. The improvement in the example over constant-gain feedback of the same two states was due entirely to the capability of freely moving the four poles of the controllable observable system, whereas with constant gain their movement was restricted. A contributing factor to the excellent performance was that only one of the five closed-loop poles could not be influenced and it was near the optimal pole position. It is reasonable to expect good performance under such fortunate circumstances. In the general case, however, a designer needs to know how much system performance is being degraded by the eigenvalues he cannot influence (those of the three canonic subsystems besides the controllable, observable one). An estimate, or bounds on J_s, would be very useful in this situation.

A variation of Ozer's method for systems with infinite time performance measure was presented. A fifth-order example

was solved, resulting in very good performance at best and marginal at worst. (An advantage of Ozer's method is that such information is available). The deviation from optimal trajectories is attributable to the small number of states being fed back. As developed in this investigation, the state weighting matrix $\underset{\sim}{Q}$ had to be positive definite. It would be helpful if this requirement could be relaxed.

Designing practical controllers for the time-varying linear regulator is much more difficult. When all the states are available the methods of Ozer or of Kleinman and Athans are feasible; variable switching times or fixed times based on the optimal gain curves should improve the results. When only outputs are available for time-varying systems, optimal controller complexity becomes excessive and suboptimal controller design must be done without the very useful canonic frame of reference. An area for further research is determining conditions for stabilizability of the time-varying linear regulator system with output feedback.

BIBLIOGRAPHY

[A-1] B. D. O. Anderson and J. B. Moore, Linear Optimal Control, Prentice-Hall, Englewood Cliffs, New Jersey, 1971.

[A-2] M. Athans and P. L. Falb, Optimal Control: An Introduction to the Theory and Its Applications, McGraw-Hill Book Company, New York, 1966.

[B-1] M. V. Bhandarkar and M. M. Fahmy, "Canonical Decomposition of Linear Time-Varying Analytic Systems," 3rd Asilomar Conf. on Circuits and Systems, pp. 177-182, 1969.

[B-2] S. P. Bingulac, "An Alternate Approach to Expanding PA + AP = -Q," IEEE Trans. on Automatic Control, Vol. AC-15, pp. 135-136, February 1970.

[D-1] F. Dellon and P. E. Sarachik, "Optimal Control of Unstable Linear Plants with Inaccessible States," IEEE Trans. on Automatic Control, Vol. AC-13, pp. 491-495, October 1968.

[F-1] J. D. Ferguson and Z. V. Rekasius, "Optimal Linear Control Systems with Incomplete State Measurements," IEEE Trans. on Automatic Control, Vol. AC-14, pp. 135-140, April 1969.

[G-1] E. A. Galperin and E. I. Dergacheva, "Optimum Stabilization of a Linear System with Incomplete Information," Automation and Remote Control, No. 8, pp. 5-15, August 1968.

[G-2] E. G. Gilbert, "Controllability and Observability in Multivariable Control Systems," SIAM J. Control, Ser. A, Vol. 1, No. 2, pp. 128-151, 1963.

[H-1] R. Hooke and T. A. Jeeves, "Direct Search Solution of Numerical and Statistical Problems," J. Assoc. Comp. Mach., Vol. 8, No. 2, pp. 212-229, April 1961.

[K-1] R. E. Kalman, "A New Approach to Linear Filtering and Prediction Theory," Journal of Basic Engineering, Trans. ASME, Series D, Vol. 82, pp. 35-45, March 1960.

[K-2] R. E. Kalman, "Contributions to the Theory of Optimal Control," Bol.Soc.Mat.Mex., Vol. 5, pp. 102-119, 1960.

[K-3] R. E. Kalman, "On the General Theory of Control Systems," Proceedings of the First International Congress of the International Federation of Automatic Controls, Vol. 1, pp. 481-492, 1960.

[K-4] R. E. Kalman, "Canonical Structure of Linear Dynamical Systems," Proc. of the Nat'l. Academy of Sciences, USA, Vol. 48, no. 4, pp. 596-600, April 1962.

[K-5] R. E. Kalman and J. E. Bertram, "Control System Analysis and Design via the "Second Method" of Lyapunov: I, Continuous time systems; II, Discrete-time systems," Trans. ASME, series D, Journal of Basic Engineering, Vol. 83, pp. 371-393, 1960.

[K-6] R. E. Kalman and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory," Journal of Basic Engineering, Trans. ASME, series D, Vol. 83, pp. 95-108, March 1961.

[K-7] D. E. Kirk, Optimal Control Thëory, An Introduction, Prentice-Hall, Englewood Cliffs, New Jersey, 1970.

[K-8] D. L. Kleinman and M. Athans, "The Design of Suboptimal Linear Time-varying Systems," IEEE Trans. on Automatic Control, Vol. AC-13, pp. 150-159, April 1968.

[K-9] D. L. Kleinman, T. Fortmann and M. Athans, "On the Design of Linear Systems with Piecewise Constant Feedback Gains," IEEE Trans. on Automatic Control, Vol. AC-13, pp. 354-361, August 1968.

[K-10] W. D. Koenigsberg and D. K. Frederick, "Output Feedback Control with Application to Unstable Linear Systems," Preprints, 1970 JACC (Atlanta, Georgia), pp. 674-682.

[L-1] W. S. Levine and M. Athans, "On the Optimal Error Regulation of a String of Moving Vehicles," Preprints, 1966 JACC (Seattle, Washington), pp. 661-667.

[L-2] D. G. Luenberger, "Observing the State of a Linear System," IEEE Trans. on Military Electronics, Vol. MIL-8, pp. 74-80, April 1964.

[L-3] D. G. Luenberger, "Observers for Multivariable Systems," IEEE Trans. on Automatic Control, Vol. AC-11, pp. 190-197, April 1966.

[M-1] F. T. Man, "Suboptimal Control of Linear Time-invariant Systems with Incomplete Feedback," IEEE Trans. on Automatic Control, Vol. AC-15, pp. 112-114, Feb. 1970.

[M-2] J. S. Meditch, "A Class of Suboptimal Linear Controls," IEEE Trans. on Automatic Control, Vol. AC-11, pp. 433-439, July 1966.

[O-1] E. Ozer, "Design on Constrained Optimal Controls for Linear Regulators with Incomplete State Feedback," E.E.Thesis, N.P.S., Monterey, California, 1969.

[P-1] J. B. Pearson and C. Y. Ding, "Compensator Design for Multivariable Systems," IEEE Trans. on Automatic Control, Vol. AC-14, pp. 130-134, April 1969.

[W-1] D. J. Wilde, Optimum Seeking Methods, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[W-2] S. Wolfenstein, Introduction to Linear Algebra and Differential Equations, Holden-Day, Inc., San Francisco, 1969.

[W-3] W. M. Wonham, "Pole Assignment in Multi-Input Controllable Linear Systems," IEEE Trans. on Automatic Control, Vol. AC-12, pp. 660-665, December 1967.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Documentation Center Cameron Station Alexandria, Virginia 22314	2
2.	Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
3.	Prof. D. E. Kirk, Code 52Ki Department of Electrical Engineering Naval Postgraduate School Monterey, California 93940	3
4.	LCDR David T. Cowdrill, USN Norfolk Naval Shipyard Portsmouth, Virginia	3
5.	Professor H. A. Titus, Code 52Ts Department of Electrical Engineering Naval Postgraduate School Monterey, California 93940	1
6.	LT Ergin Özer, Turkish Navy 98. Sokak, No. 11-2 Göztepe, Izmir Turkey	1
7.	Dr. S. J. Yakowitz, Code 52Ya Department of Electrical Engineering Naval Postgraduate School Monterey, California 93940	1
8.	LCDR R. R. Owens, USN USS Truxton (DLGN-36) FPO San Francisco, California	1

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3 REPORT TITLE

Methods of Suboptimal Control for Linear Regulators Using Output Feedback

4 DESCRIPTIVE NOTES (Type of report and, inclusive dates)

Electrical Engineer; March 1971

5 AUTHOR(S) (First name, middle initial, last name)

David Thomas Cowdrill

6 REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
March 1971	89	29

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10 DISTRIBUTION STATEMENT

Approved for public release;distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

Methods are discussed for optimal and suboptimal control of linear regulator systems employing controllers which use only accessible states and which can be easily realized. The conditions required for stability of such systems are shown and an algorithm is introduced for determining the elements of a stabilizing constant-gain matrix. This matrix provides an initial point for a technique which sub-optimizes the performance of systems with infinite-time performance measures. The concept of partial canonic state feedback is introduced in which a Luenberger observer obtains missing canonic state information which is combined with the plant output to form a feedback vector. Also given is an extension of an existing algorithm for determining piecewise-constant gains to include switching times as variable parameters. Examples are given to numerically illustrate the concepts and results.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Linear Regulator						
Optimal Control						
Suboptimal Control						
Suboptimal Linear Regulator						
Incomplete State Feedback						
Output Feedback						

22 SEP 78 S 1 0 1 6 0

128140
Thesis
C756955 Cowdrill
c.1 Methods of subopti-
mal control for linear
regulators using out-
put feedback.
22 SEP 78 S 1 0 1 6 0

128140
Thesis
C756955 Cowdrill
c.1 Methods of subopti-
mal control for linear
regulators using out-
put feedback.

thesC756955
Methods of suboptimal control for linear
3 2768 001 02148 8
DUDLEY KNOX LIBRARY